Rotated Einstein-Podolsky-Rosen States and Their Properties

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Outline

States with perfect correlation

- continuous system: EPR state (1935)

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(x_1 - x_2 + x_0)p} dp$$

- finite-dimensional system: Bohm state (1951)

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Outline

Comparison of EPR state and Bohm state

- common:
  - bi-partite systems.
  - perfectly correlated.
- differences:
  - Bohm state is well-defined;
    EPR state is not well-defined.
  - all states with perfect correlation on $\mathbb{C}^2 \otimes \mathbb{C}^2$ are unitarily equivalent to Bohm state;
    for continuous system it is unknown.

Outline

Problems:

- a well-defined formulation of EPR state
- rotated EPR states
- entanglement properties
- measurement of individual particles

Outline

EPR states

The observable algebra of a particle is given by $\mathcal{A}(\mathbb{R}^2)$:

$$\mathcal{A}(\mathbb{R}^2) = \overline{\left\{ \sum_{j=1}^{n} c_j W(u_j) \;\middle|\; u_j = (a_j, b_j) \in \mathbb{R}^2 \right\}}$$

In Schrödinger representation $\mathcal{L}(\mathbb{R})$
$W(u)$ can be represented as

$$\pi_S(W(u)) = e^{i(a\hat{q}+b\hat{p})}$$

with

$$\begin{aligned} \hat{q} &: \quad \text{position operator} \\ \hat{p} &: \quad \text{momentum operator} \end{aligned}$$

EPR states

$W(u)$ satisfies
the **canonical commutation relation (CCR)**:

$$W(u)W(v) = e^{-i\sigma(u,v)/2} W(u+v) \qquad (1)$$

with

$$\sigma(u,v) = u_1 v_2 - u_2 v_1, \quad u, v \in \mathbb{R}^2. \qquad (2)$$

EPR states

In $\mathcal{A}(\mathbb{R}^2)$ we have $*$-operation:

$$W(u)^* = W(-u)$$

An **observable** associated with an measurement procedure corresponds to an element $A \in \mathcal{A}(\mathbb{R}^2)$ with

$$A = A^*.$$

A **state** ω of a particle is given by a linear positive functional on $\mathcal{A}(\mathbb{R}^2)$ with norm one:

$$\omega \; : \; \mathcal{A}(\mathbb{R}^2) \to \mathbb{C}.$$

EPR states

Let $\mathcal{A} = \mathcal{B} = \mathcal{A}(\mathbb{R}^2)$.
The observable algebra of two particles is then given by

$$\mathcal{A} \otimes \mathcal{B} \cong \mathcal{A}(\mathbb{R}^4)$$

where the elements $W(u) \in \mathcal{A}(\mathbb{R}^4)$ satisfy CCR similarly as (1) with

$$\sigma_2 = \sigma \oplus \sigma.$$

In Schrödinger representation $\mathcal{L}(\mathbb{R}^2)$:

$$\begin{aligned} \pi_S(W(a, 0, -a, 0)) &= e^{ia(\hat{q}_1 - \hat{q}_2)}, \quad \text{relative position} \\ \pi_S(W(0, b, 0, b)) &= e^{ib(\hat{p}_1 + \hat{p}_2)}, \quad \text{total momentum.} \end{aligned}$$

EPR states

Halvorson (2000):
ω_{epr} is a state on $\mathcal{A}(\mathbb{R}^4)$ that assigns
a dispersion-free value λ_0 to $\hat{q}_1 - \hat{q}_2$ and
a dispersion-free value μ_0 to $\hat{p}_1 + \hat{p}_2$.

$$\omega_{epr}(W(a,b,c,d)) = \delta(a+c)\delta(b-d)e^{i(a\lambda_0 + b\mu_0)}. \qquad (3)$$

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3. $\omega(A)$: the **expectation value** of A in ω.

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4. The **dynamics** of a physical system $\mathcal{A}$ is a *-automorphism of $\mathcal{A}$.

C^*-algebraic formulation of quantum mechanics

Examples:

1. Systems with finite dimensions:

$$\begin{aligned} \mathcal{A} &= M_n \\ \rho(A) &= \mathrm{tr}(\rho A), \quad \rho \leq 0, \quad \mathrm{tr}\rho = 1 \end{aligned}$$

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2. systems of infinitely many dimensions:

$$\begin{aligned} \mathcal{A} &= \mathcal{B}(\mathcal{L}(\mathbb{R})) \\ \rho(A) &= \langle \Omega, A\Omega \rangle, \quad \Omega \in \mathcal{L}(\mathbb{R}), \quad \|\Omega\| = 1 \end{aligned}$$

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EPR representations

Halvorson (2000):
Let $l_2(\mathbb{R}^2)$ be the Hilbert space of square-summable functions from $\mathbb{R}^2$ to $\mathbb{C}$:

$$
\begin{aligned}
f &: \mathbb{R}^2 \to \mathbb{C}, \\
\langle f|g\rangle &= \sum \overline{f(\lambda,\mu)} g(\lambda,\mu), \\
\|f\| &= (\sum |f(x)|^2)^{1/2}.
\end{aligned}
$$

EPR representations

$\xi_{(\lambda,\mu)}$: the characteristic function of the set $\{(\lambda,\mu)\}$:

$$\xi_{(\lambda,\mu)}(x,y) = \begin{cases} 1 & (x,y) = (\lambda,\mu) \\ 0 & (x,y) \neq (\lambda,\mu) \end{cases}$$

Define a linear mapping $\pi_{epr} : \mathcal{A}(\mathbb{R}^4) \to \mathcal{B}(l_2(\mathbb{R}^2))$

$$\begin{aligned} \pi_{epr}(W(a,0,-a,0))\xi_{(\lambda,\mu)} &= e^{ia\lambda}\xi_{(\lambda,\mu)}, \\ \pi_{epr}(W(0,b/2,0,-b/2))\xi_{(\lambda,\mu)} &= \xi_{(\lambda-b,\mu)}, \\ \pi_{epr}(W(c/2,0,c/2,0))\xi_{(\lambda,\mu)} &= \xi_{(\lambda,\mu+c)}, \\ \pi_{epr}(W(0,d,0,d))\xi_{(\lambda,\mu)} &= e^{id\mu}\xi_{(\lambda,\mu)}. \end{aligned}$$

EPR representation

$(l_2(\mathbb{R}^2), \pi_{epr}, \xi_{(0,0)})$ has properties:

- $\xi_{(0,0)}$ is cyclic for $\pi(\mathcal{A} \otimes \mathcal{B})$:

$$\overline{\{\pi(\mathcal{A} \otimes \mathcal{B})\xi_{(0,0)}\}} = l_2(\mathbb{R}^2)$$

- ω_{epr} is represented as a vector state $\xi_{(0,0)}$:

$$\omega_{epr}(A) = (\xi_{(0,0)}, \pi_{epr}(A)\xi_{(0,0)}), \quad \forall A \in \mathcal{A} \otimes \mathcal{B}$$

$(l_2(\mathbb{R}^2), \pi_{epr}, \xi_{(0,0)})$ is called a **GNS** representation of ω_{epr}.

Rotated EPR states

Bohr (1935): in EPR states $(\hat{q}, \hat{p})$ cab be replaced $(\hat{Q}, \hat{P})$

$$\hat{Q}_1 = \hat{q}_1 \cos\theta + \hat{q}_2 \sin\theta, \qquad \hat{Q}_2 = -\hat{q}_1 \sin\theta + \hat{q}_2 \cos\theta,$$
$$\hat{P}_1 = \hat{p}_1 \cos\theta + \hat{p}_2 \sin\theta, \qquad \hat{P}_2 = -\hat{p}_1 \sin\theta + \hat{p}_2 \cos\theta.$$

1. This corresponds to a rotation R in phase space.
2. The commutation relation remains, i.e.,

$$[\hat{Q}_j, \hat{P}_j] = i, \quad j = 1, 2.$$

Rotated EPR states

Remark of Bohr (1935) + ω_{epr} by Halvorson (2000)

$\Rightarrow$ rotated EPR states (Huang, 2007)

Rotated EPR states

Introduce a new basis $E = [u_1, v_1, u_2, v_2]$ for $\mathbb{R}^4$,

$$u_1 = (\cos\theta, 0, \sin\theta, 0) \qquad v_1 = (0, \cos\theta, 0, \sin\theta),$$
$$u_2 = (-\sin\theta, 0, \cos\theta, 0), \qquad v_2 = (0, -\sin\theta, 0, \cos\theta).$$

$$W(u_1) \Leftrightarrow \hat{Q}_1 \qquad W(v_1) \Leftrightarrow \hat{P}_1$$
$$W(u_2) \Leftrightarrow \hat{Q}_2, \qquad W(v_2) \Leftrightarrow \hat{P}_2.$$

Rotated EPR states

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Rotated EPR states

- A state which assigns share values to $W(u_1)$ and $W(v_2)$ has the same perfect correlation as the EPR states.
- Let $(a, b, c, d)_\theta$ denote the coordinate vector of an element x with respect to the new ordered basis $E = [u_1, v_1, u_2, v_2]$.
- A **rotated EPR state** ω_θ such that $\hat{Q}_1$ and $\hat{P}_2$ have the sharp value 0 can then be defined as

$$\omega_\theta(W(a, b, c, d)_\theta) = \delta_{b,0}\delta_{c,0} \tag{4}$$

Rotated EPR states

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- τ_R is a *-automorphism,

$$\tau_R(W(u)) = W(Ru)$$

Rotated EPR states

Define $\pi_\theta : \mathcal{A}(\mathbb{R}^4) \to \mathcal{B}(l_2(\mathbb{R}^2))$:

$$\begin{aligned}
\pi_\theta(W(a,0,0,0))\xi_{(\lambda,\mu)} &= e^{ia\cos\theta\lambda}\xi_{(\lambda,\mu-a\sin\theta)},\\
\pi_\theta(W(0,b,0,0))\xi_{(\lambda,\mu)} &= e^{-ib\sin\theta\mu}\xi_{(\lambda-b\cos\theta,\mu)},\\
\pi_\theta(W(0,0,c,0))\xi_{(\lambda,\mu)} &= e^{ic\sin\theta\lambda}\xi_{(\lambda,\mu+c\cos\theta)},\\
\pi_\theta(W(0,0,0,d))\xi_{(\lambda,\mu)} &= e^{id\cos\theta\mu}\xi_{(\lambda-d\sin\theta,\mu)}.
\end{aligned}$$

$\xi_{(0,0)}$ has the following properties:

$$\begin{aligned}
\overline{\{\pi_\theta(\mathcal{A}\otimes\mathcal{B})\xi_{(0,0)}\}} &= l_2(\mathbb{R}^2)\\
\omega_\theta(A) &= (\xi_{(0,0)}, \pi_\theta(A)\xi_{(0,0)})
\end{aligned}$$

$(l_2(\mathbb{R}^2), \pi_\theta, \xi_{(0,0)})$: **GNS representation** of ω_θ

Rotated EPR representation

Property I:

$\xi_{(0,0)}$ has the entanglement property:

$$\overline{\{\pi_\theta(\mathcal{A})\xi_{(0,0)}\}} = \overline{\{\pi_\theta(\mathcal{B})\xi_{(0,0)}\}} = l_2(\mathbb{R}^2).$$

Rotated EPR representation

Property (II):

Halvorson (2000):
ω_{epr} maximally violate Bell's inequalities.

Here: similar arguments as Halvorson (2000),
ω_θ maximally violate Bell's inequalities.

Rotated EPR states

Property (II):

ω_θ has the perfect correlation:

If the outcome of one measurement on one subsystem is obtained, then the outcome of some measurement on the other subsystem can be predicted with certain.

Rotated EPR states

Property (IV):

No information about individual particle can be obtained!

- The operator $\hat{q}_1$ does not exist.
  Due to the weak discontinuity of $\omega_\theta(W(a,0,0,0))$

$$\begin{aligned} \omega_\theta(W(a,0,0,0)) &= \langle \xi_{(0,0)}, \pi(W(a,0,0,0))\xi_{(0,0)} \rangle \\ &= \begin{cases} 1, & a = 0 \\ 0, & a \neq 0 \end{cases}, \end{aligned}$$

  the limit does not exist!

$$\lim_{a \to 0} \frac{W(a,0,0,0) - \mathbb{I}}{a} \xi_{(0,0)} =: i\hat{q}\, \xi_{(0,0)}$$

- Similarly, $\hat{q}_2$, $\hat{p}_1$ and $\hat{p}_2$ does not exist.

Rotated EPR states

- Keyl, Schlingemann, and Werner (2003):
  In an EPR states the probability of finding a particle at infinity is one!
- Halvorson (2004):
  In any representation where the position operator has eigenstates, there is no momentum operator, and vice versa.

Rotated EPR states

Property (V):

The uncertainty principle implies that two representations π_θ and $\pi_{\theta'}$ are not unitarily equivalent if $\theta \neq \theta' + n\pi$ or $\theta \neq \theta' + (n + 1/2)\pi$, i.e., there is no unitary operator U on $l_2(\mathbb{R}^2)$ such that

$$\begin{aligned} \pi_\theta(W(x)) &= U^\dagger \pi_{\theta'}(W(x)) U \\ \xi_{(0,0)} &= U \xi_{(0,0)} \end{aligned}$$

Rotated EPR states

- For finite systems $M_n \otimes M_n$:
  states with perfect correlation:

  $$\Phi = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |e_j f_j\rangle, \quad \{e_j\}, \{f_j\} \text{ ONB for } \mathbb{C}^n$$

  All states satisfying perfect correlation can transfered into each other by a unitary operator on the same vector space $\mathbb{C}^n \otimes \mathbb{C}^n$.
- For infinite systems $\mathcal{A}(\mathbb{R}^n)$:
  there exists non-unitarily equivalent states satisfying perfect correlation.
  $\Rightarrow$ New entanglement phenomenon.

Perfect correlation

$\mathcal{A}_1$, $\mathcal{A}_2$: two independent C^*-algebras
ρ: state $\mathcal{A}_1 \otimes \mathcal{A}_2$
Perfect correlation (Werner, 1999): $\forall A \in \mathcal{A}_1$, $\exists B \in \mathcal{A}_2$

$$\rho((A - B)(A^* - B^*)) = \rho((A^* - B^*)(A - B)) = 0.$$

B: an EPR-double of A

$(\mathcal{H}, \pi, \Omega)$: the GNS representation of ρ, $\rho(A) = (\Omega, \pi(A)\Omega)$

$$\pi(A)\Omega = \pi(B)\Omega, \quad \pi(A)^*\Omega = \pi(B)^*\Omega.$$

Perfect correlation

For ω_θ with $(l_2(\mathbb{R}^2), \pi_\theta, \xi_{(0,0)})$ define an anti-unitary operator J:

$$J[c\chi_{(\lambda\cos\theta,\mu\sin\theta)}] = e^{-i\lambda\mu\cos 2\theta}\,\overline{c}\,\chi_{(-\lambda\cos\theta,-\mu\sin\theta)}$$

The only EPR-double of $W(a,b,0,0) \in \mathcal{A}$ is given by

$$\begin{aligned} B &= \exp(i(a\lambda_0/\cos\theta - b\mu_0/\sin\theta)) \\ &\quad \times W(0,0,-a\sin\theta/\cos\theta, b\cos\theta/\sin\theta) \in \mathcal{B}. \end{aligned}$$

$\Rightarrow \omega_\theta$ satisfies the perfect correlation.

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2. No information of individual particle from ω_θ can be obtained!
3. ω_θ non-unitarily equivalent to $\omega_{\theta'}$
4. Another construction ω_ϕ:

$$\omega_\phi(W(a,b,c,d)_\phi) = \delta_{b,0}\delta_{c,0}e^{i(a\lambda_0 + d\mu_0)}$$

$$\begin{aligned} s_1 &= (\cos\phi, 0, 0, \sin\phi) & t_1 &= (0, \cos\phi, -\sin\phi, 0), \\ s_2 &= (-\sin\phi, 0, 0, \cos\phi), & t_2 &= (0, -\sin\phi, -\cos\phi, 0). \end{aligned}$$

Thank you for your attention!