Stokes Phenomenon and Non-Perturbative Effects in Matrix Models

Chi-Hsien Yeh
National Center for Theoretical Sciences (NCTS)
with
Hirotaka Irie (YITP) and **Chuan-Tsung Chan** (Tunghai Univ.)

Base on arXiv:1011.5745, 1109.2598, 1206.2351, …

Perturbation vs. Non-perturbation

When we calculate physics quantities in perturbative expansion, we may need to answer several questions:

1. Is the perturbative series convergent?

2. What are the contributions of non-perturbative parts?

3. Is the perturbative vacuum stable?

Outline

- Perturbation and Non-perturbation in Matrix Model
- Stokes Phenomenon and Riemann-Hilbert Problem
- Landscape in Matrix Model
- Conclusion and Future Topics

Introduction

Perturbation and Non-perturbation in Matrix Model

Example: Matrix model

Matrix Models (N: size of Matrix)

$$\mathcal{Z}_{MM} = \int dM e^{-Ntr(V(M))}$$

M: NxN Hermitian Matrix

String Theory

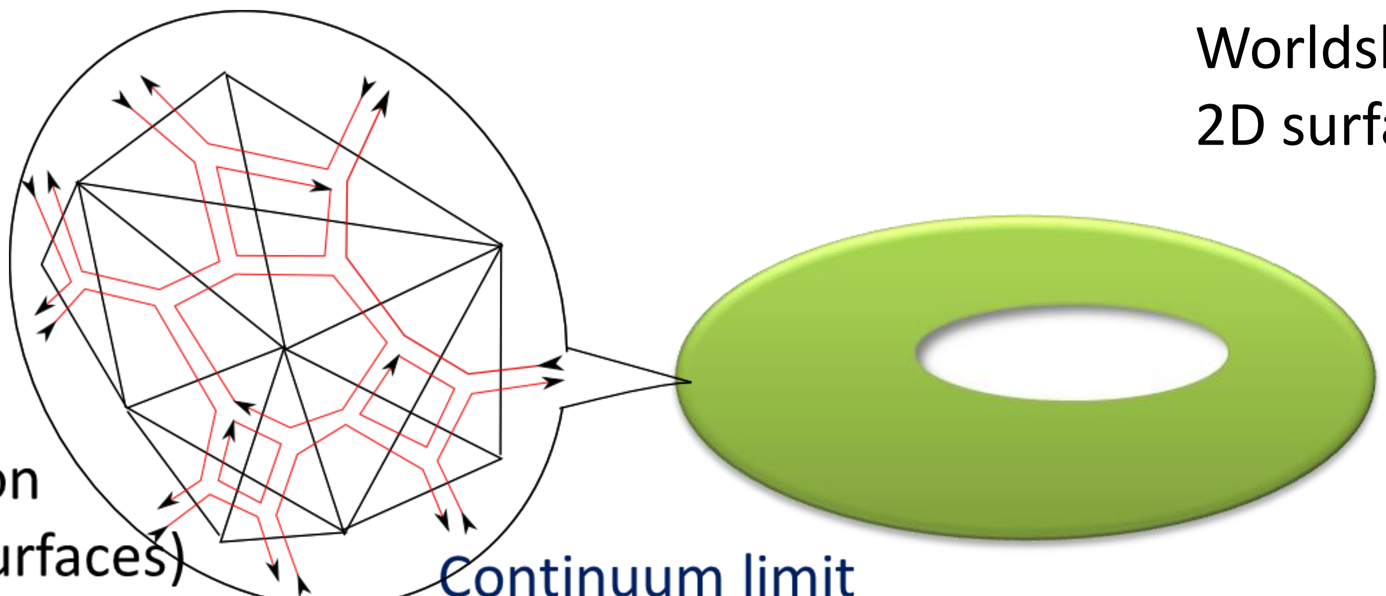
$$\mathcal{Z} \approx \sum_{topologies} \int \mathcal{D}g \ldots e^{\int d^2\sigma \sqrt{g} \ldots}$$

$$N^{-1} = g_{str}$$

$\Downarrow$ Feynman Diagram

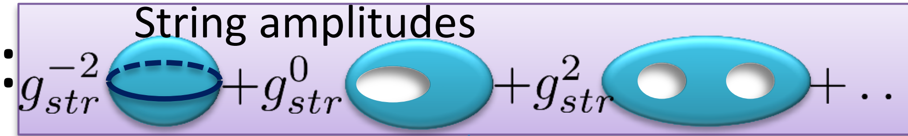
$\Downarrow$ Worldsheet 2D surfaces

Triangulation (Random surfaces)



Free Energy in Matrix model

- **Perturbative expansions:**



$$\mathcal{F} = ln \mathcal{Z}_{MM} \cong \sum_{n=0}^{\infty} N^{2-2n} \mathcal{F}_n \quad \Longleftrightarrow \quad \sum_{n=0}^{\infty} g_{str}^{2n-2} \mathcal{F}_n$$

n: genus

- **Non-perturbative parts**:

$$(\mathcal{F} - \sum_{n=0}^{\infty} g_{str}^{2n-2} \mathcal{F}_n) \equiv \mathcal{F}_{\text{nonpert.}} = \sum_I \theta_I e^{\frac{-1}{g_{str}} \mathcal{F}^{(I)}} + \dots$$

- Determine $\mathcal{F}^{(I)}$: Instanton calculations
- Determine θ_I : Stokes phenomenon

Values of θ_I tell us which instanton effect is important

Details of Matrix model

Diagonalization: $U^\dagger M U = diag(\alpha_1, \alpha_2, \cdots, \alpha_N)$

$$\mathcal{Z}_{MM} = \int dM e^{-N tr(V(M))} \Rightarrow \int d^N\alpha \prod_{i<j}(\alpha_i - \alpha_j)^2 e^{-N\sum_i V(\alpha_i)}$$

N-body problem in the potential V

Orthogonal Polynomial Method:

$$\int d\alpha e^{-V(\alpha)} P_n(\alpha) P_m(\alpha) = h_n \delta_{n,m} \qquad P_n(\alpha) = \alpha^n + \cdots$$

$$\prod_{i<j}(\alpha_i - \alpha_j) = det(\alpha_i^{j-1}) = det(P_{j-1}(\alpha_i)) \Rightarrow \mathcal{Z}_{MM} = N! \prod_{n=0}^{N-1} h_n$$

Q: How do we solve h_n ?

Continuum Limit

$$
\begin{aligned}
\alpha P_n &= \sum_{m=0}^{n+1} A_{nm} P_m \\
\frac{d}{d\alpha} P_n &= \sum_{m=0}^{n-1} B_{nm} P_m
\end{aligned}
$$

$P_n(\alpha) \to \psi(t;\zeta)$
with $n \to t,\ \alpha \to \zeta$

$N \to \infty$

Baker-Akhiezer function system

$$
\begin{aligned}
\zeta \psi(t;\zeta) &= \mathbf{P}(t;\partial)\psi(t;\zeta) \\
\frac{d}{d\zeta}\psi(t;\zeta) &= \mathbf{Q}(t;\partial)\psi(t;\zeta)
\end{aligned}
$$

$\mathbf{P}(t;\partial)$,$\mathbf{Q}(t;\partial)$ are p-th and q-th order differential operator in $\partial = g_{str}\partial_t$

$$\int d\alpha e^{-V(\alpha)} \alpha P_n P_m \neq 0,\ |m-n| \leq 1$$

$$\int d\alpha \frac{d}{d\alpha}(P_n P_m e^{-V(\alpha)}) = 0$$

These relations give the recursion relations of h_n.
It is called string equation.

String Equation:

$$[\mathbf{P}, \mathbf{Q}] = g_{str}\mathbf{1}$$

The commutator ($\mathbf{P}$,$\mathbf{Q}$) gives us the differential equation of coefficients which are in ($\mathbf{P}$,$\mathbf{Q}$)

Summary of (p,q) Minimal String

- After we take continuum limit, the different potential forms of matrix models are correspond to different Baker-Akhiezer systems with (**P**,**Q**) pairs.

- The theory in continuum side is understood as **2D gravity couple to (p,q) minimal CFT matter fields** with central charges:

$$c_{\text{matter}}^{(p,q)} = 1 - 6\frac{(p-q)^2}{pq}, \quad \text{p and q coprime}$$

- **String equation**: Nonlinear Differential equations what we obtain in continuous limit: Painlevé equations, …

- Ex:(p,q)=(2,3), pure gravity: Painlevé Ⅰ equation

$$\partial^2 u + 6(u^2 + t) = 0; \;\; \mathbf{P} = \partial^2 + u(t), \mathbf{Q} = \partial^3 + \cdots$$

Method of Analysis

Stokes Phenomenon

and

Riemann-Hilbert Problem

Stokes phenomenon of Airy function

Airy function: $(\frac{d^2}{d\zeta^2} - \zeta)Ai(\zeta) = 0$

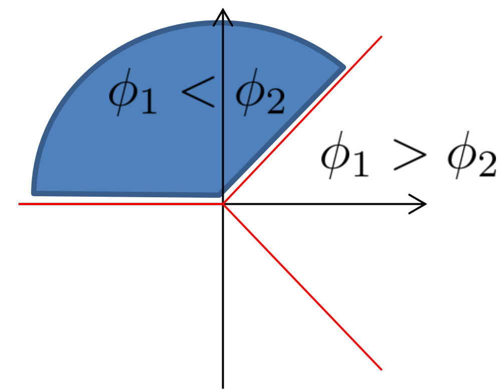
Two WKB solutions :

$$\phi_1 = \frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}} \text{ and } \phi_2 = \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}$$

Different asymptotic expansions in different regions

$\zeta \to \infty$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}[1 + \cdots]$$



$\zeta \to -\infty$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}[1 + \cdots] + \mathbf{i}\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}[1 + \cdots]$$

Stokes phenomenon

Solutions have different asymptotic expansions in different regions: **Stokes sectors**

The difference is the coefficient of sub-dominated term: **Stokes multipliers**

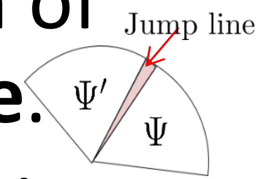
What can we learn?

- **Stokes Matrix** describes the difference:

$$\Psi' = (a' \quad b') = (a \quad b) \begin{pmatrix} 1 & 0 \\ s_{21} & 1 \end{pmatrix} \equiv \Psi \mathbf{S}$$

Ai function case:
$a = 0, b = 1, s_{21} = i$

- When the intersection region shrink to line, the Stokes multipliers give the jump information of function Ψ, this line is called **anti-Stokes line**.



- Giving a ODE and finding solutions, the solutions have jump behavior (Stokes Phenomenon)
- When we want to solve the coefficient in the ODE system. Can we solve it from the information of Stokes matrices (jump information) ?
- Yes! It is called **Inverse Monodromy Method or Riemann-Hilbert Problem**

p x p ODE system in Matrix Models

- For a given (p,q) matrix model, we have a correspondent Baker-Akhiezer systems :

$$\begin{aligned} \zeta\psi(t;\zeta) &= \mathbf{P}(t;\partial)\psi(t;\zeta) = (\partial^p + \cdots)\psi(t;\zeta) \\ g_{str}\frac{d}{d\zeta}\psi(t;\zeta) &= \mathbf{Q}(t;\partial)\psi(t;\zeta) = (\partial^q + \cdots)\psi(t;\zeta) \end{aligned}$$

- We can rewrite the BA system to be a p x p ODE system with p different dominations:

$$\zeta \to \lambda^p \quad \vec{\psi} \sim \left(\psi, \psi', \ldots, \psi^{(p-1)}\right)^T \qquad \Psi := \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_p\right);\ p \times p \text{ matrix}$$

- We find how to relate ∂ by function of t and λ , so we obtain a p x p ODE system in λ plane:

$$g_{str}\frac{\partial}{\partial\lambda}\Psi(t;\lambda) = \mathcal{Q}(t;\lambda)\Psi(t;\lambda) = (p\Omega^q\lambda^{p+q-1} + \cdots)\Psi(t;\lambda)$$

$$\Omega = diag(1, \omega, \omega^2, \cdots, \omega^{p-1});\ \omega^p = 1$$

Stokes Phenomenon in Matrix Models

- Consider pxp ODE system with matrix function $\Psi(\lambda)$ ($\zeta = \lambda^p$):

$$\frac{\partial}{\partial\lambda}\Psi(t;\lambda) = \mathcal{Q}(t;\lambda)\Psi(t;\lambda) = (p\Omega^q\lambda^{r-1} + \cdots)\Psi(t;\lambda)$$

$$r \text{ is called Poincaré index, here } r = p + q$$

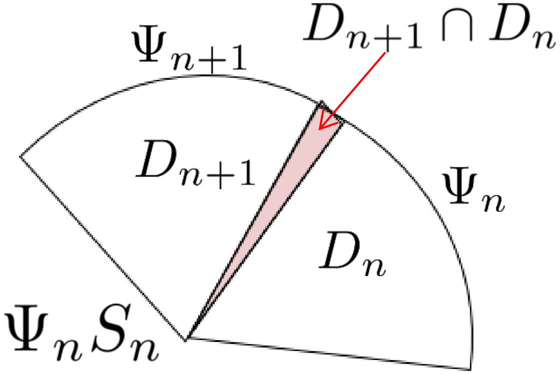
- Matrix function $\Psi(\lambda)$ has a formal asymptotic solution in $\lambda \to \infty$.

$$\begin{aligned}\Psi_{asym} &= Z(\lambda)e^{\varphi(\lambda)} \\ &= (1 + \frac{Z_1}{\lambda} + \cdots)e^{\Omega^q\lambda^{p+q}+\cdots}\end{aligned}$$

- Asymptotic expansions are only applied in specific λ angular domains, and exact solutions in each domain differ by a constant matrix (*Stokes matrices*):

$$\Psi_n(\lambda) \approx \Psi_{asym}(\lambda);\ \lambda \to \infty, \lambda \in D_n$$

$$D_n = \{\lambda \in \mathbb{C}; \frac{n-1}{rp}\pi < arg\lambda < \frac{n+p}{rp}\pi\}$$



$$\Psi_{n+1} = \Psi_n S_n$$

Algebra Relations of Stokes Matrices

- **Z_p-symmetry condition**: $\lambda \to \omega\lambda$

$$S_{n+2r} = \Gamma^{-1} S_n \Gamma, \ \ (n = 0, 1, \cdots, 2rp-1)$$

$$\Gamma = (\delta_{j,i+1} + \delta_{i,p}\delta_{j,1})_{1\le i,j\le p}$$

- **Monodromy condition**: $\lambda \to e^{2\pi i}\lambda$

$$S_0 S_1 S_2 \cdots S_{2rp-1} = e^{i\pi(p-1)} \mathbf{1}_p$$

- **Hermiticity condition**: $\lambda \to \lambda^*$

$$S_n^* = \Delta\Gamma S^{-1}_{(2r-1)p-n} \Gamma^{-1} \Delta, \ \ (n = 0, 1, \cdots, 2rp-1)$$

$$\Delta = (\delta_{i+j,p+1})_{1\le i,j\le p}$$

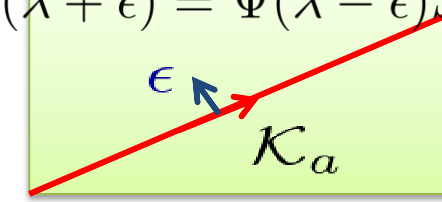
- **Multi-cut boundary condition**: Branch cuts in the λ plane are relative to the eigenvalue distributions in Matrix model. They give the additional constraint for Stokes matrices

Riemann-Hilbert Problem

- <u>Finding an analytic function having a prescribed jump across a curve</u>
- Jump Information: **Stoke matrices**
- Curves $\mathcal{K}$: **Deift-Zhou network** (as anti-Stoke lines)
- Analytic function Z(λ) are determined from its jump behavior: $Z_{\pm} = \lim_{\epsilon \to 0} Z(\lambda \pm \epsilon)$

$$Z_+ = Z_- G;\ \ G = e^{\varphi(\lambda)} S_a e^{-\varphi(\lambda)}$$

$$\Psi(\lambda+\epsilon) = \Psi(\lambda-\epsilon) S_a$$



- The solving Ψ problem becomes to a integral problem:

$$Z(\lambda) = I_p + \int_{\mathcal{K}} \frac{d\xi}{2\pi i} \frac{Z_-(\xi)(G(\xi) - I_p)}{\xi - \lambda}$$

I_p is $p \times p$ identity matrix

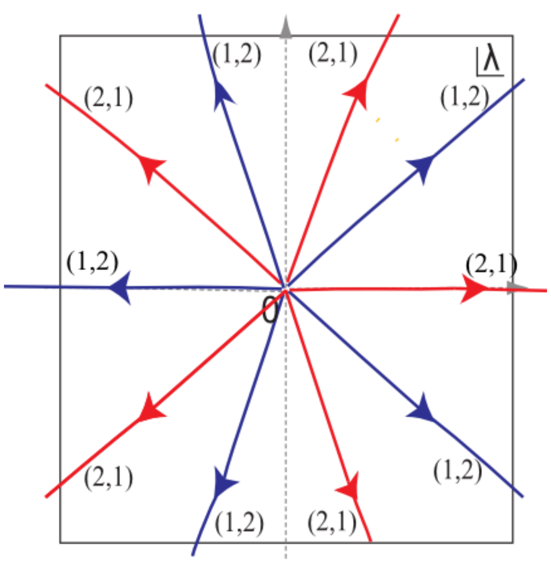
- The solution of u(t) is given from Z(λ)

Deift-Zhou network

- The main contributions of integral contour for solution Ψ are steepest descend lines:

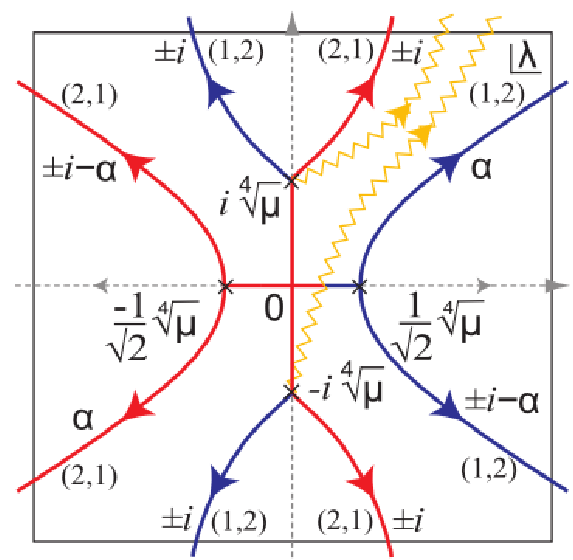
$$\mathbf{Im}(\varphi(\lambda)) = \mathbf{Im}(\varphi(\lambda^*));\ \partial_\lambda \varphi(\lambda^*) = 0$$

- Network can be deformed from anti-Stokes lines
- Ex: (p,q)=(2,3)



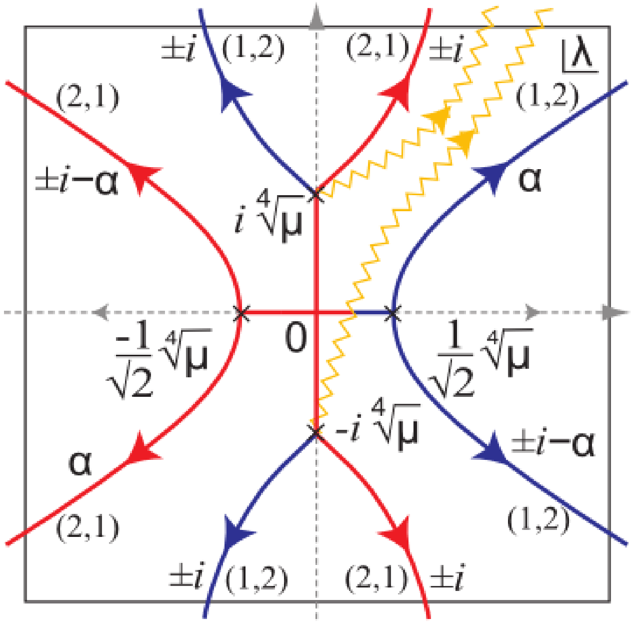
$\lambda \to \infty$

Include the solutions of Stokes multipliers



Information on Deift-Zhou network

The weight in Each lines are the solutions of Stokes multipliers which are non-zero elements of Stokes matrices. Blue lines are the line with non-zero s_{12}*, and Red lines are the line with non-zero* s_{21}*.*



◆**Behavior of Z(λ) is given by Deift-Zhou network:**

- **Perturbative part of free energy**: It comes from the branch cut structure of φ (λ).
- **Non-perturbative part of free energy**: it comes from the integral around saddle points.

Sketch of Inverse Monodromy Method

Give (p,q) **Baker-Akhiezer** system

↓ Solve

$$\frac{\partial}{\partial\lambda}\Psi(t;\lambda) = \mathcal{Q}(t;\lambda)\Psi(t;\lambda)$$

↓ $\text{Expand around } \lambda \to \infty$

$$\Psi(\lambda) = Z(\lambda)e^{\varphi(\lambda)}$$

↓ Solve

$$\textbf{Stokes matrices } S_n$$

↓ Draw

D-Z network along
saddle points of φ (λ)

→

$$\text{Along countour } \mathcal{K}_a$$

$$\Psi(\lambda+\epsilon) = \Psi(\lambda-\epsilon)S_a$$

↓ Calculate RH

$$Z(\lambda) = I_p + \int_{\mathcal{K}} \frac{d\xi}{2\pi i}\frac{Z_-(\xi)(G(\xi)-I_p)}{\xi-\lambda}$$

$$G = e^{\varphi(\lambda)}S_a e^{-\varphi(\lambda)}; \lambda \in \mathcal{K}_a \subset \mathcal{K}$$

↓ Calculate u(t)

$$u(t) = -4\lim_{\lambda\to\infty}\lambda^2 Z(\lambda)$$

↓ Calculate Free Energy

$$u = \partial^2 F$$

Applications of non-perturbative researches

Structure of vacua (Landscape): meta-stable vacuum, true vacuum, and decay rate

Example ((p,q)=(2,5) Yang-Lee Edge)

- (**P**,**Q**) pairs:

$$\mathbf{P}(\partial;t) = \partial^2 + u(t)$$
$$\mathbf{Q}(\partial;t) = \partial^5 + v_1(t)\partial^3 + v_2(t)\partial^2 + v_3(t)\partial + v_4(t)$$

$$[\mathbf{P}(\partial;t), \mathbf{Q}(\partial;t)] = g_{str}$$

- Classical solution of φ（λ ）:**Chebyshev background**

$$\mathbf{P} \to T_p(z), \mathbf{Q} \to T_q(z), z = \frac{\partial}{\mu^{1/4}}$$

Chebyshev polynomial:
$T_n(cos\theta) = cos(n\theta)$

$$\varphi_{mst}(\lambda) = \int \mathbf{dP}\ \mathbf{Q}(\mathbf{P}) = \mu^{\frac{7}{4}} \int^{\frac{\lambda^2}{\sqrt{\mu}}} dz \sqrt{\frac{T_5(z)+1}{2}}$$

- String equation is:

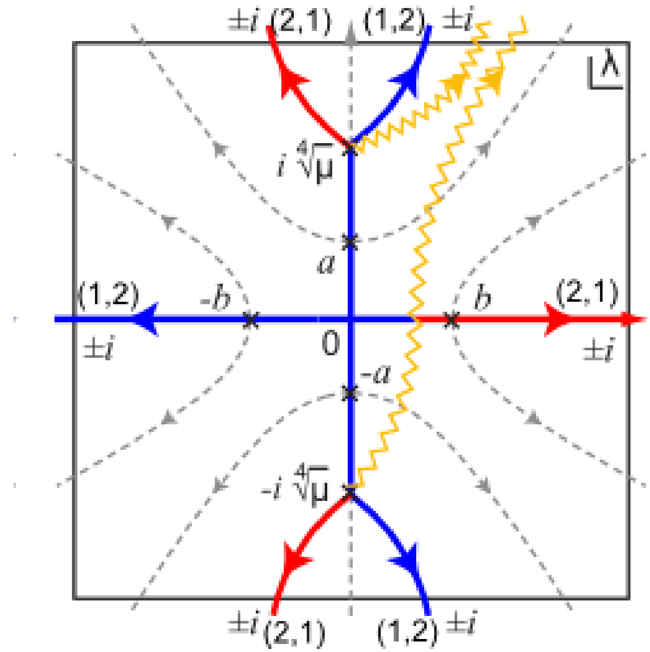
$$\partial^4 u + 20u\partial^2 u + 10(\partial u)^2 - 40\mu u + 40u^3 = -16t$$

Chebyshev background: Meta-stable vacuum

Deift-Zhou network (Yang-Lee Edge (p,q)=(2,5)):

$$\varphi_{mst}(\lambda) = \mu^{\frac{7}{4}} \int^{\frac{\lambda^2}{\sqrt{\mu}}} dz \sqrt{\frac{T_5(z)+1}{2}}$$

$$T_5(z) = 16z^5 - 20z^3 + 5z$$



Large Instanton: **meta-stable**

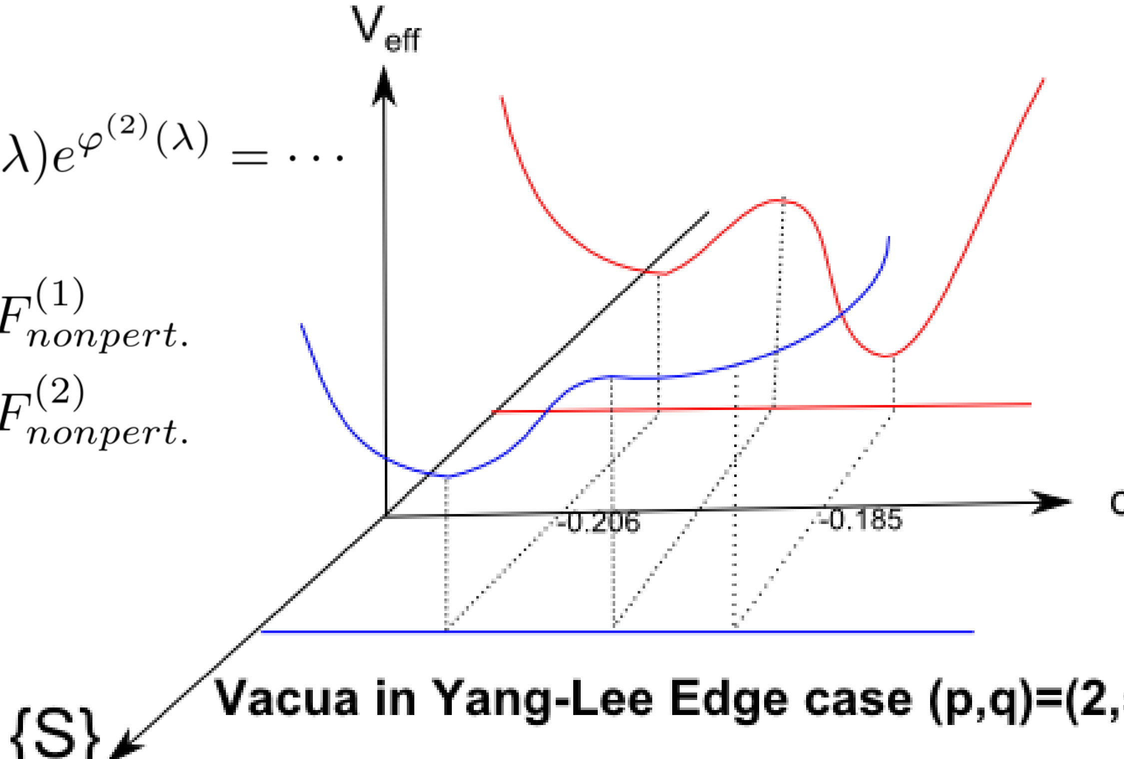
$$\mathcal{F}_{nonpert} = \mp \frac{\sqrt{g_{str}/2}\exp\left[+\frac{10}{21 g_{str}}\sqrt{2(5-\sqrt{5})}\mu^{\frac{7}{4}}\right]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}+1)^{\frac{5}{2}}\mu^{\frac{7}{8}}}} + \cdots$$

Landscape of Matrix Model

- For fixed Stokes matrices, the solution of $\Psi(\lambda)$ is unique. But the factorization ways can be different!

$$\Psi(\lambda) = Z_{(1)}(\lambda) e^{\varphi^{(1)}(\lambda)} = Z_{(2)}(\lambda) e^{\varphi^{(2)}(\lambda)} = \cdots$$

$$\begin{aligned} \varphi^{(1)}(\lambda) &\quad \to \quad F = F^{(1)}_{pert.} + F^{(1)}_{nonpert.} \\ \varphi^{(2)}(\lambda) &\quad \to \quad F = F^{(2)}_{pert.} + F^{(2)}_{nonpert.} \\ &\quad \vdots \end{aligned}$$

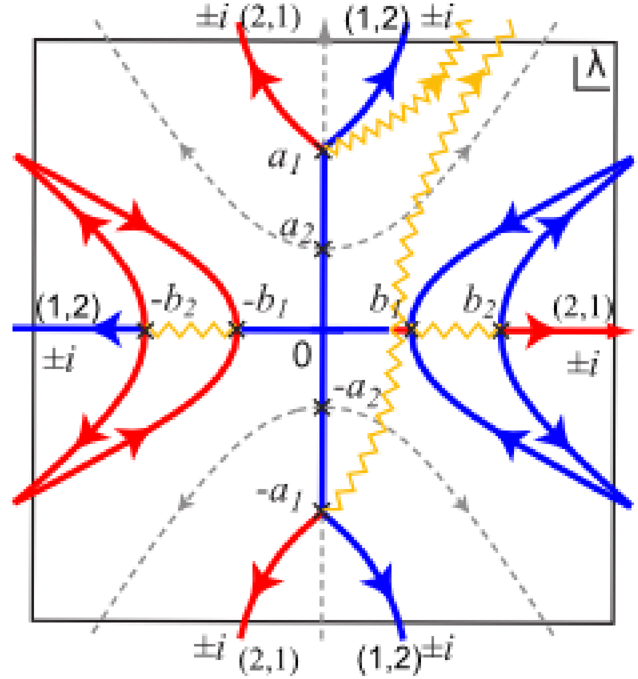


Vacua in Yang-Lee Edge case (p,q)=(2,5)

True Vacuum

Deift-Zhou network (Yang-Lee Edge (p,q)=(2,5)) :

$$\varphi_{tv}(\lambda) = \mu^{\frac{7}{4}} \int^{\frac{\lambda^2}{\sqrt{\mu}}} d\wp(\wp - \frac{5}{2}c)\sqrt{4\wp^3 - g_2(c)\wp - g_3(c)}$$



u(μ) in true vacuum

$$u_{pert}(\mu) \simeq -\sqrt{\mu}(\wp(\omega_A) + \wp(\omega_B) - \wp(\omega_A + \omega_B) + c)$$

No large instanton condition: (c~−0.184963725)

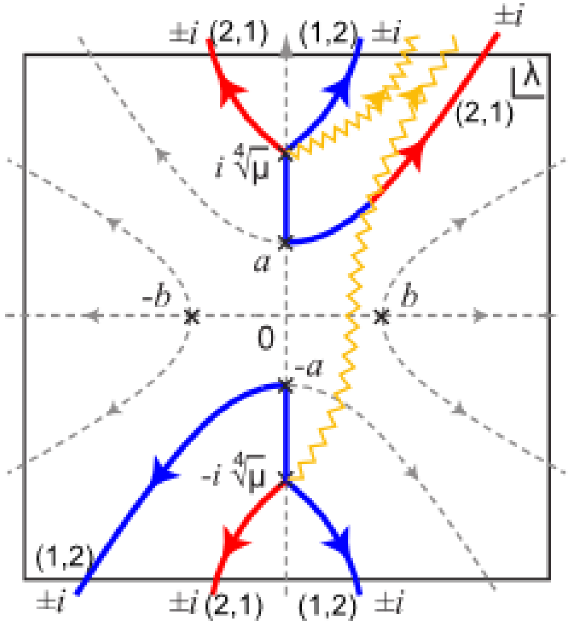
$$[\frac{4g_2(c)}{5}\eta(\omega_B(c)) - \frac{6g_3(c)\omega_B(c)}{5}]\frac{5}{2}c = \frac{6g_3(c)}{7}\eta(\omega_B(c)) - \frac{g_2^2(c)\omega_B(c)}{21}$$

$\wp$: Weierstrass elliptic function, $-\wp' = \eta$

$\omega_A \& \omega_B$: Weierstrass half period along the A-cycle and B-cycle

Decay Rate

Deift-Zhou network: contour deformation as Coleman's method for bounce solution



Decay rate: Imaginary part of free energy

$$\mathcal{F} \simeq \mathcal{F}_{pert} \mp \frac{i}{2} \frac{\sqrt{g_{str}/2}\exp\left[-\frac{10}{21 g_{str}}\sqrt{2(5+\sqrt{5})}\mu^{\frac{7}{4}}\right]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}-1)^{\frac{5}{2}}\mu^{\frac{7}{8}}}}$$

A Summary of Landscape in Yang-Lee Edge

- Large instanson implies *chebyshev vacuum* is meta-stable: $\mathbf{P} \to T_p(z), \mathbf{Q} \to T_q(z)$
- Based on the RH problem researches, we find the vacuum spaces are governed by *Weierstrass elliptic function*: $\mathbf{P} \to \wp, \mathbf{Q} \to (\wp - \frac{5}{2}c)\sqrt{4\wp^3 - g_2(c)\wp - g_3(c)}$
- The vacua are described by parameter c:

$$\text{ex: } c = -0.206 \text{ Chebyshev background, } c = -0.185 \text{ true vacuum}$$

- The different perturbative vacuums can be understood as: same Baker-Akhiezer function has different factorization forms with fix jump information (λ around infinity)

Goals on Future

- Generalize these results to all (p,q) minimal string theories
- Find a principle to determine the position of true vacuum
- Understand the concept of duality in Matrix models
(Ex: T-duality or S-duality, etc.)
- Find applications for other systems: String theory, Condensed Matter Physics, Mathematical Physics, etc.
- Answer the questions: Why do we live in this special vacuum? Is it stable or meta-stable?

Thanks For Your Attention

“The important thing is not to stop questioning.”

-Albert Einstein