The quantum phase transition and universality from numerical corner transfer matrices

National Center for Theoretical Sciences (NCTS)
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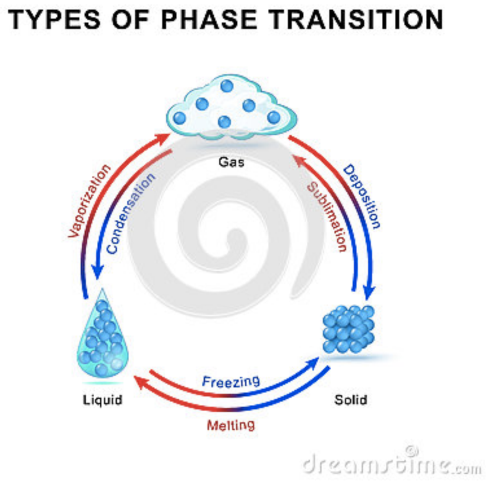
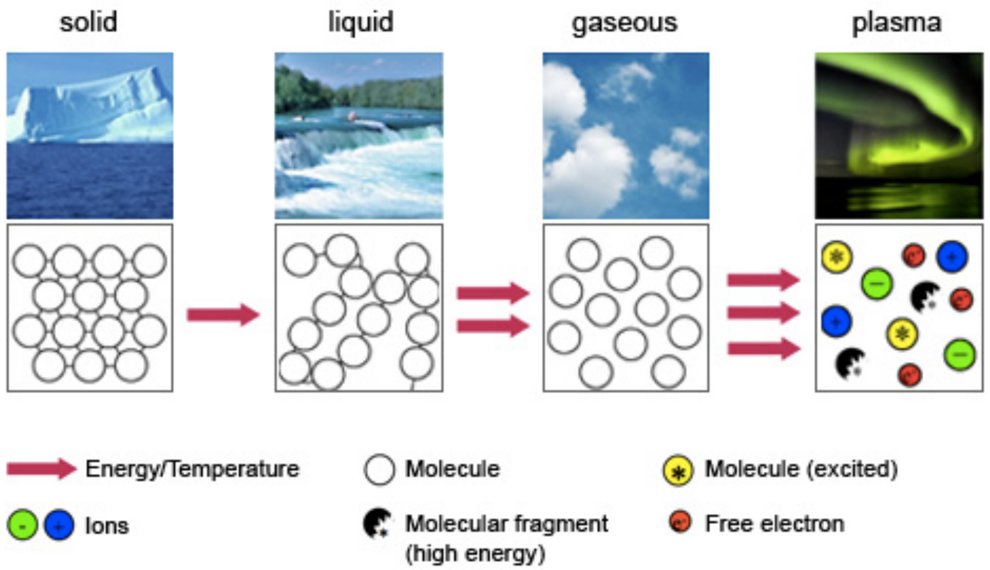
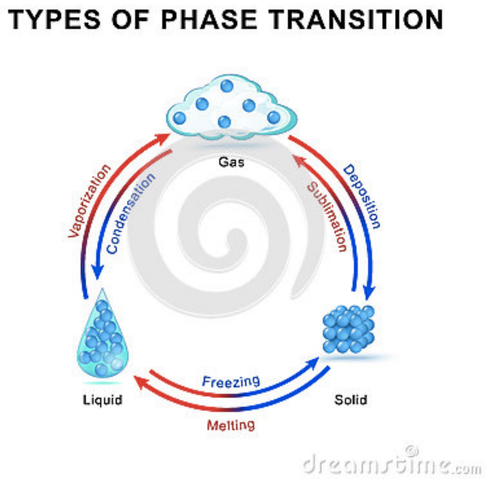
Motivation


TYPES OF PHASE TRANSITION
Gas
Vaporization
Condensation
Deposition
Sublimation
Liquid
Solid
Freezing
Melting
dreamstime.com

Motivation

- Different phases - **rich world**

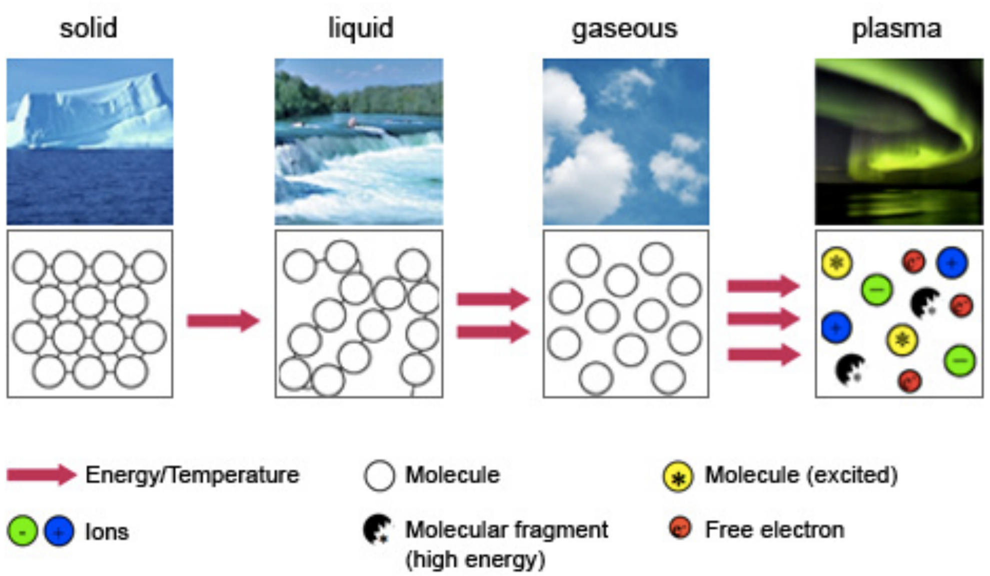


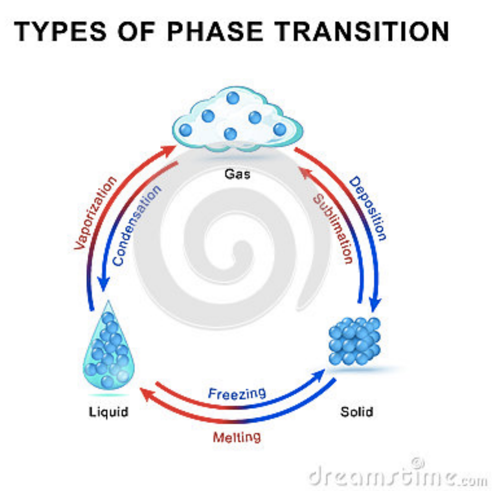


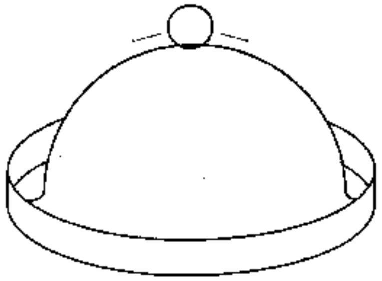
Motivation

- Different phases - **rich world**
- Why the different phases exist
  - **Symmetry breaking theory**

[Landau]







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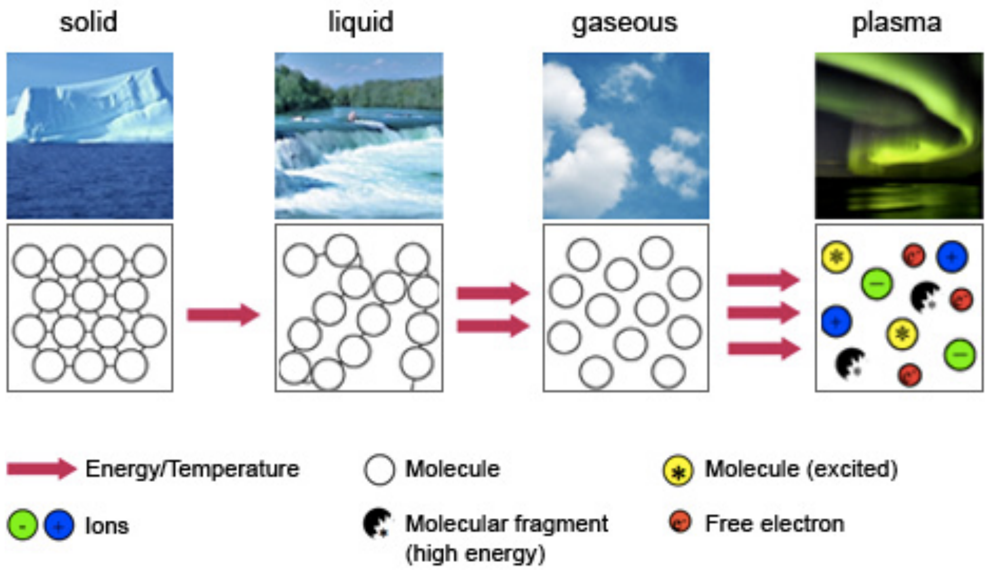
- Magnets: rotation symmetry breaking



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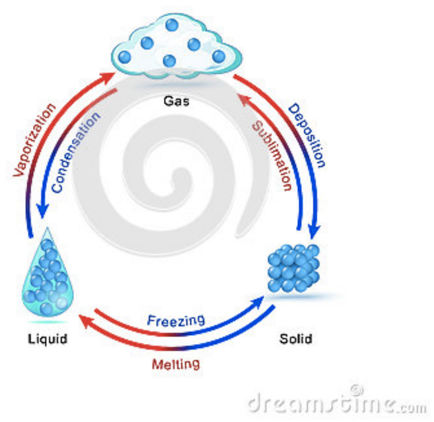

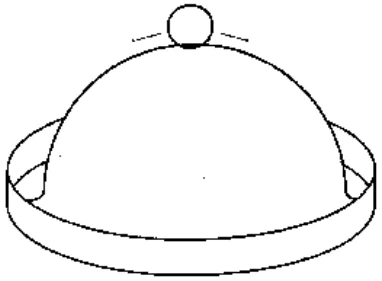

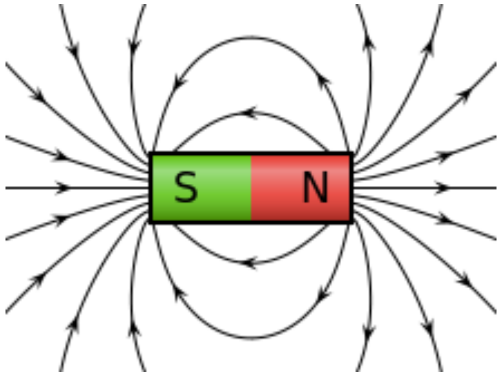

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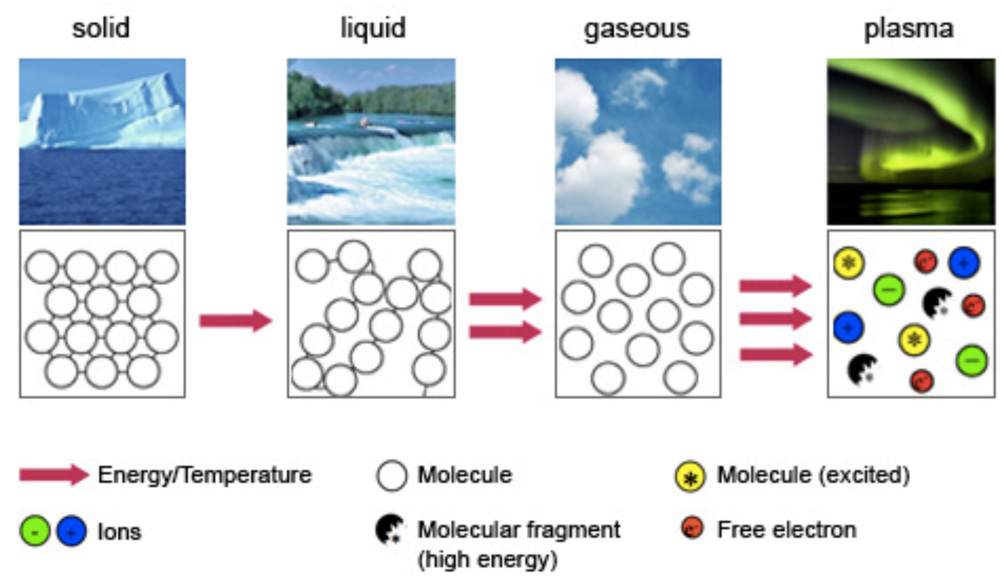

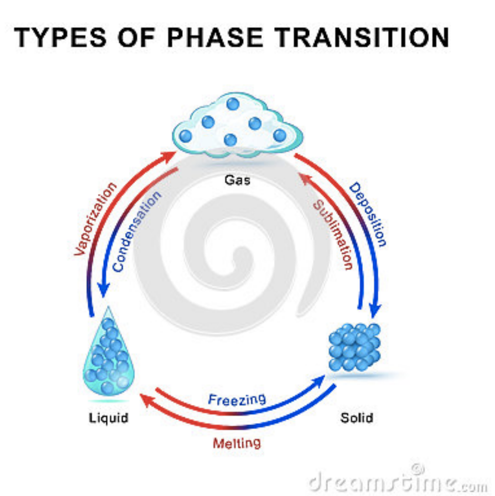

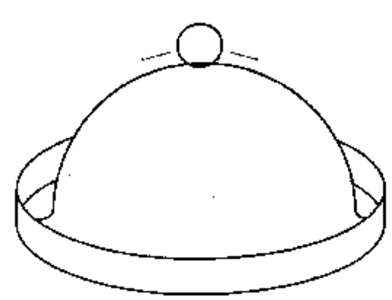

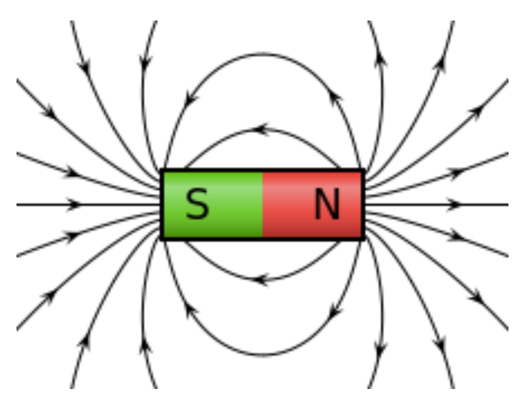

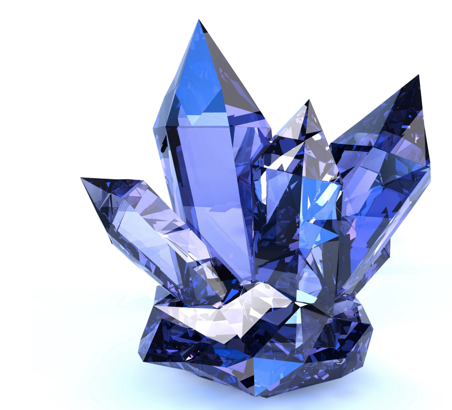

Motivation

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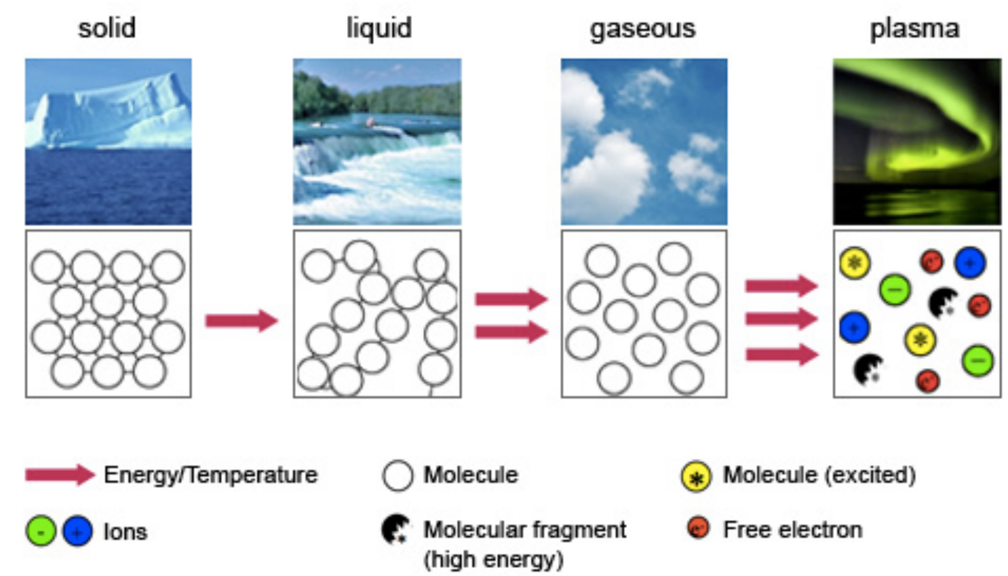
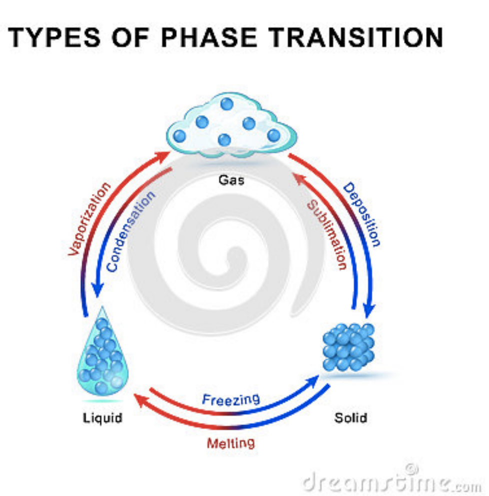
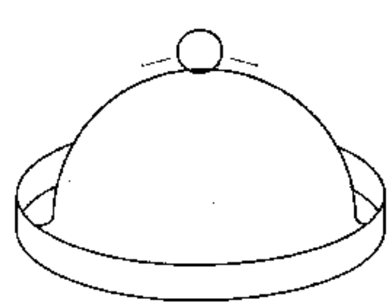
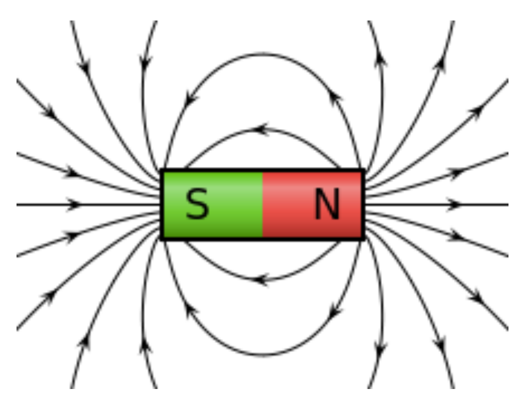
- Crystals: translation symmetry breaking…
- **Local order parameters** distinguish different phases



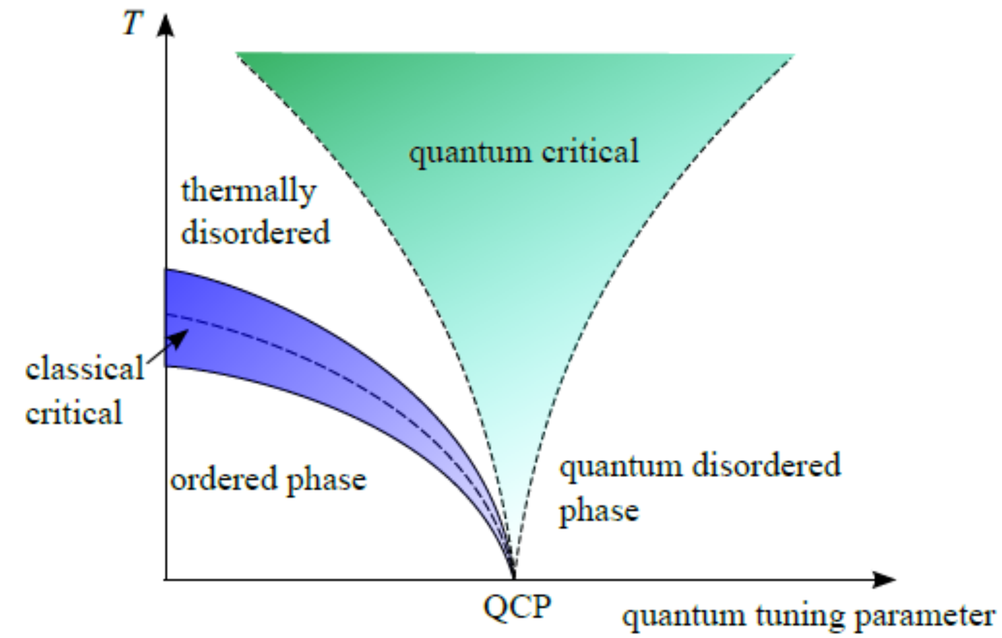






Motivation

- Phase transition:
  - Type of transition
  - Characteristic properties:
    Symmetry, Order parameters, Critical points,
    Critical exponents and Universality classes,…



[Sachdev,'11]

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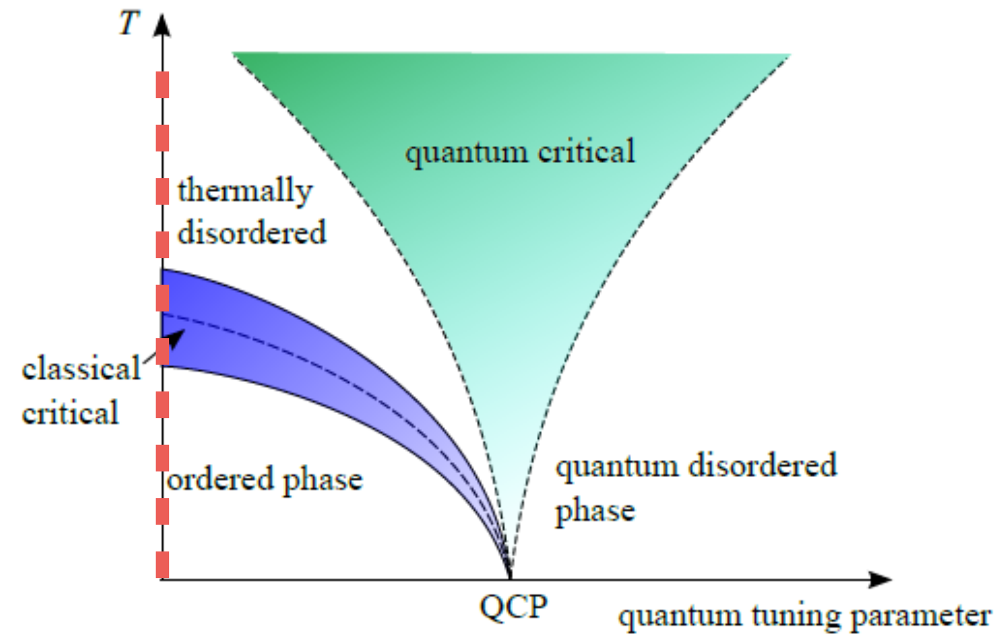
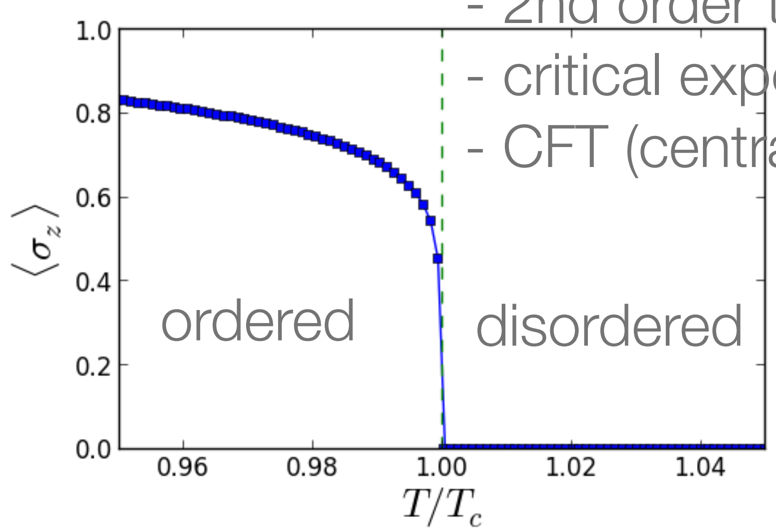


[Sachdev,'11]

- **Example**:
  2D classical Ising model

$$H = -\sum_i (\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x)$$

h=0



- 2nd order transition
- critical exponents
- CFT (central charge c=1/2)

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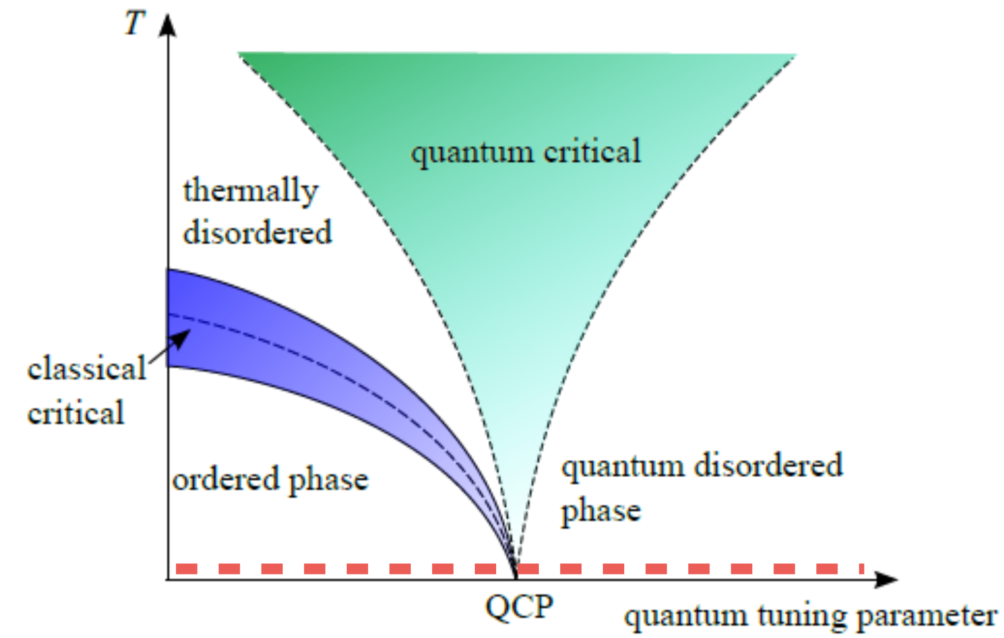

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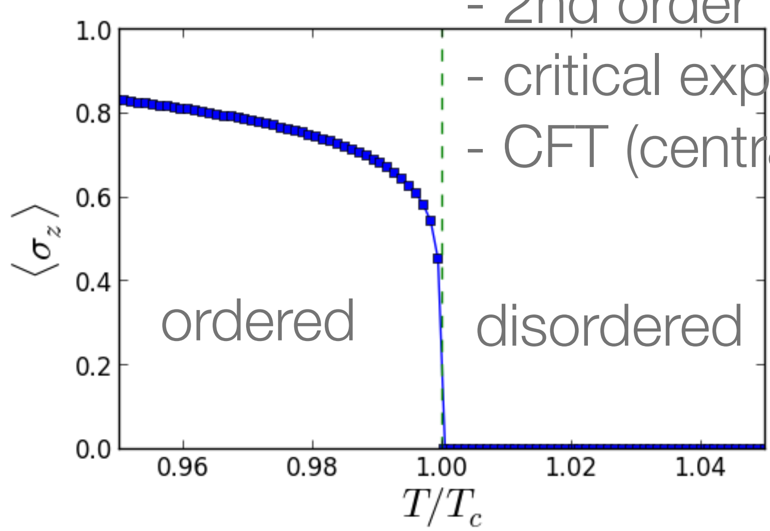
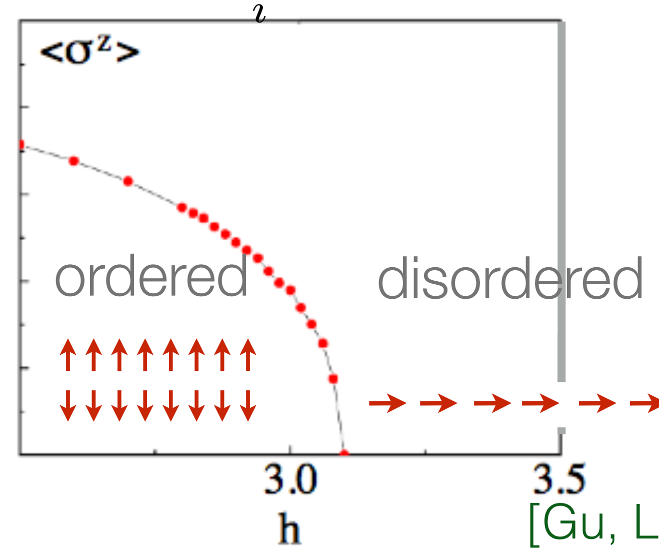


- **Example**:
  2D quantum Ising model with Transverse field h

$$H = -\sum_i (\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x)$$



[Gu, Levin,& Wen' 08]

Motivation

- To study quantum many-body system
- The Hilbert space grows exponentially with system size $\mathcal{H} \sim d^N$
- To efficient simulation (polynomial in memory and time)
- To study various Hamiltonians (e.g. Bosons and Fermions) and measure physical properties and observables

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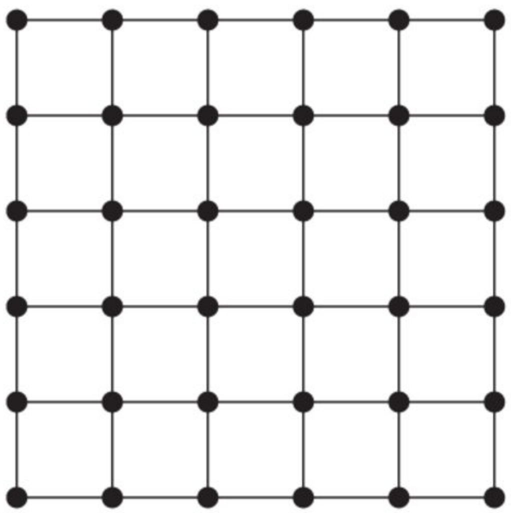
Find the ground state (approximation)
Measure physical observables (approximation)
exact diagonalization (ED),
Density matrix renormalization group (DMRG)
Tensor network state (simple update, full update)+
Tensor network algorithm (PEPS,TRG,SRG,HOTRG,TNR,.....)
....

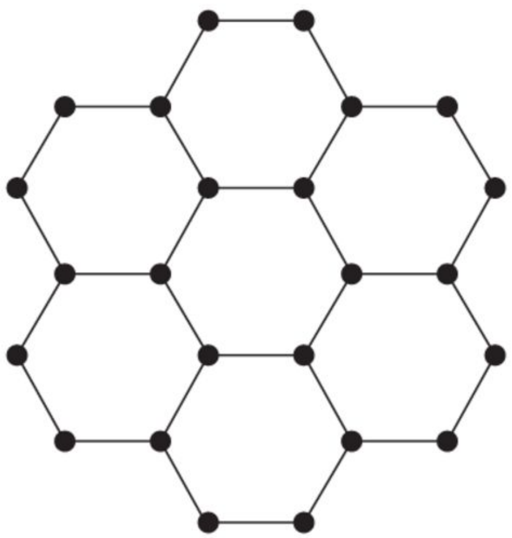
Outline

- Introduction
  - Tensor network state
  - corner tensor
  - Symmetry protected topological order phases
- The fingerprints of universal physics are encoded holographically in numerical CTMs and CTs.
  - classical and quantum Ising (quantum-classical correspondence)
  - deformed symmetry protected topological order phase
  - 2d quantum XXZ mofrl
- Quantum state renormalization in 2D using corner tensors
  - chiral topological PEPS
- Summary

A typical problem

- We are given:
  - A lattice with N sites
  - On each site a C^d Hilbert space
  - A quantum Hamiltonian





- The most general state: $$|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_N} C_{s_1, s_2, \ldots, s_N} |s_1, s_2, \ldots, s_N\rangle$$

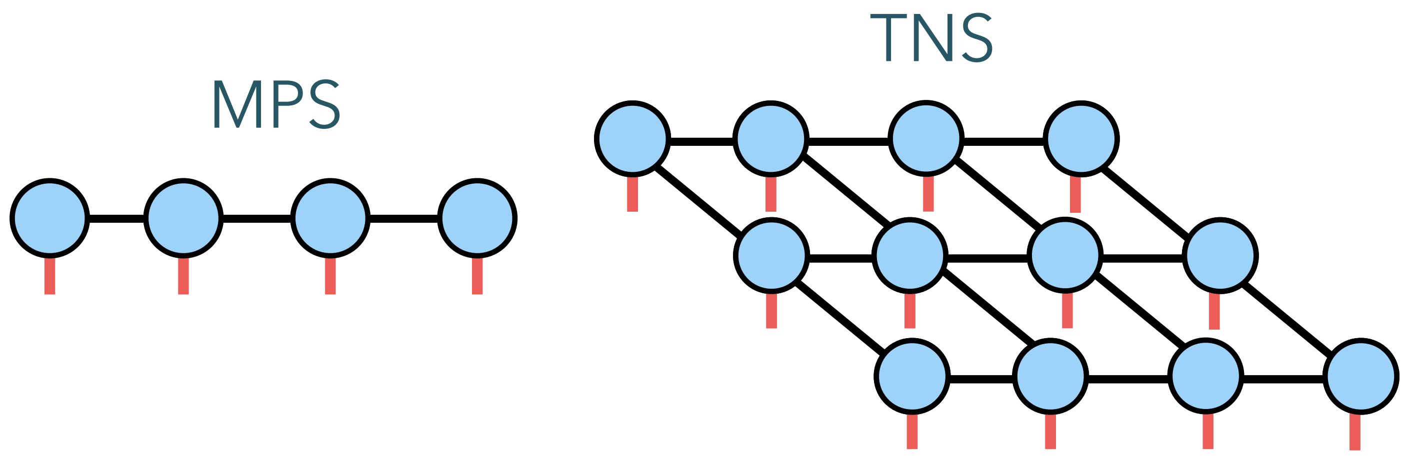
- Exponentially large number of states: d^N

Can we do better?

Matrix/Tensor Product States

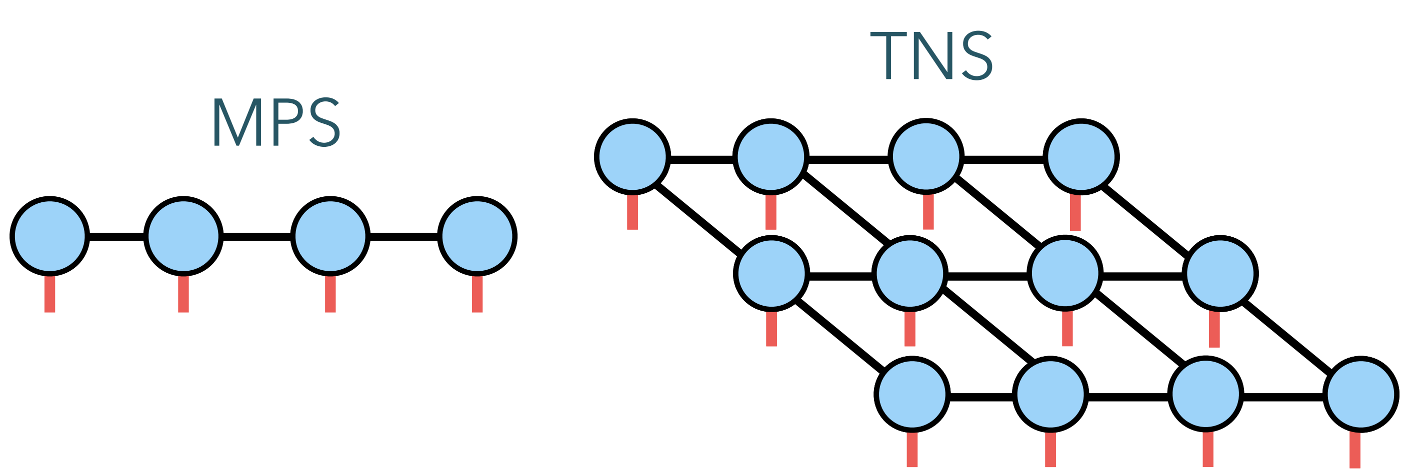
Matrix/Tensor Product States

- The numerical implementation for finding the ground states of spin systems are based on the matrix/tensor product states.



Matrix/Tensor Product States

- The numerical implementation for finding the ground states of spin systems are based on the matrix/tensor product states.
- These states can be understood from a series of Schmidt (bi-partite) decomposition. It is QIS inspired.



Matrix/Tensor Product States

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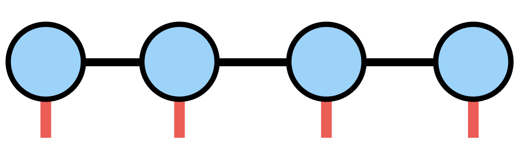
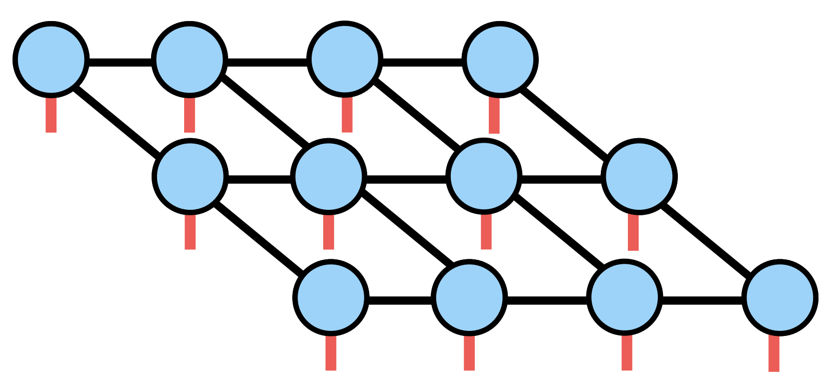
- These states can be understood from a series of Schmidt (bi-partite) decomposition. It is QIS inspired.
- The ground state is approximated by the relevance of entanglement.

MPS



TNS

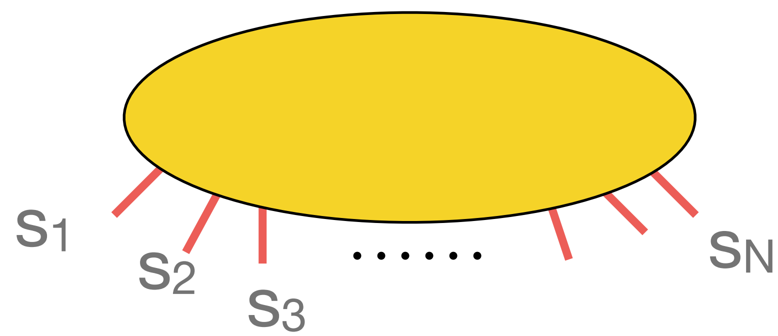


Graphical representation

- We have to deal with an N index tensor!

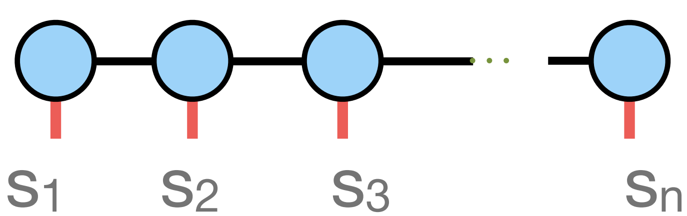
$$|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_N} C_{s_1, s_2, \ldots, s_N} |s_1, s_2, \ldots, s_N\rangle$$

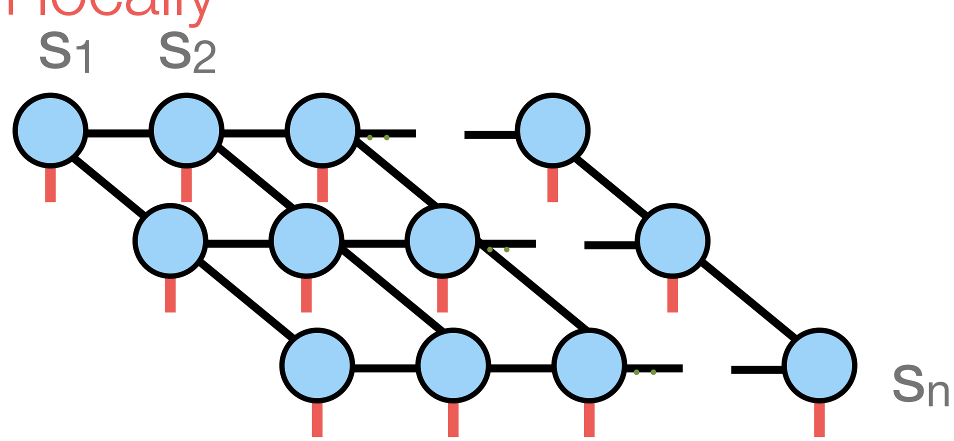
$$C_{s_1, s_2, \ldots, s_N} =$$



- Is there a good way to represent it?

Break the wave function locally



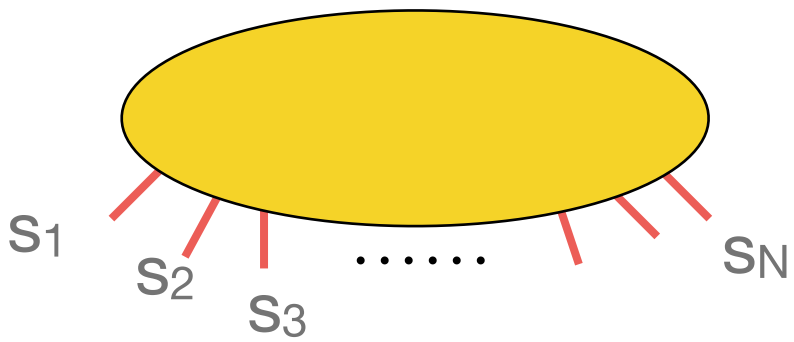


$$|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_N} C_{s_1, s_2, \ldots, s_N} |s_1, s_2, \ldots, s_N\rangle$$

d-level systems

Tensor (multidimensional array of complex numbers)

$$C_{s_1, s_2, \ldots, s_N}$$



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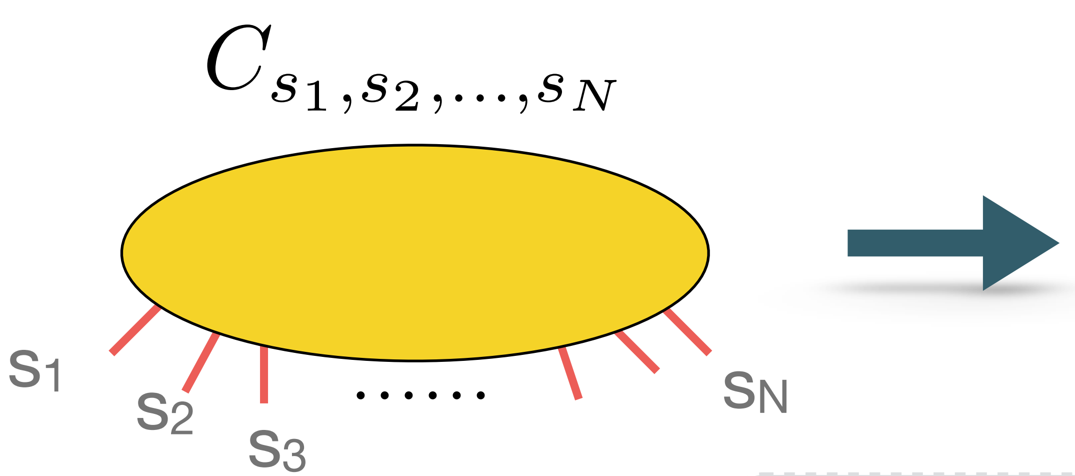
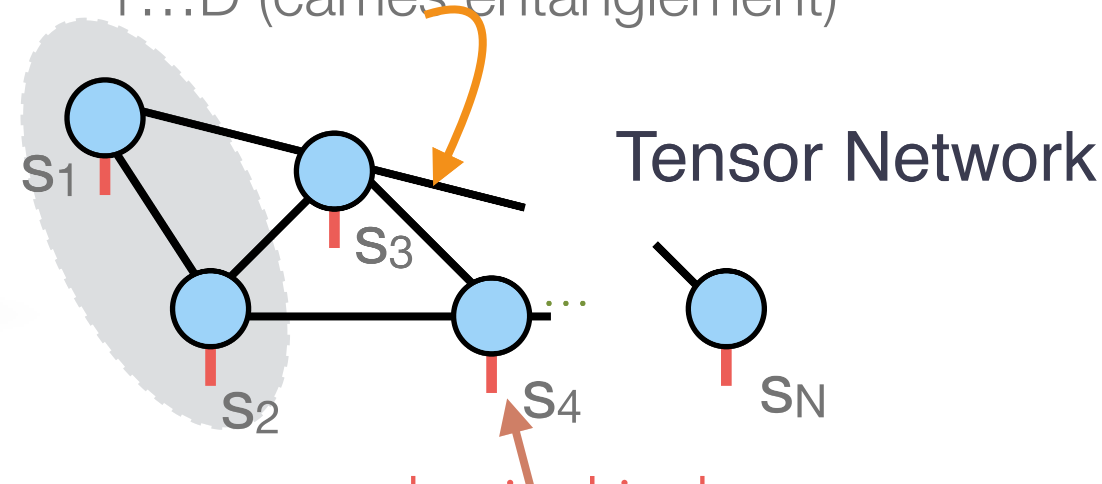


Break the wavefunction locally

summed ancillary **bond index** 1…D (carries entanglement)



physical index 1,…,d (local basis)

$$C^{s_1, s_2}_{\alpha\gamma\lambda} = \sum_{\beta} A^{s_1}_{\alpha\beta} B^{s_2}_{\beta\gamma\lambda}$$

$$|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_N} C_{s_1, s_2, \ldots, s_N} |s_1, s_2, \ldots, s_N\rangle$$

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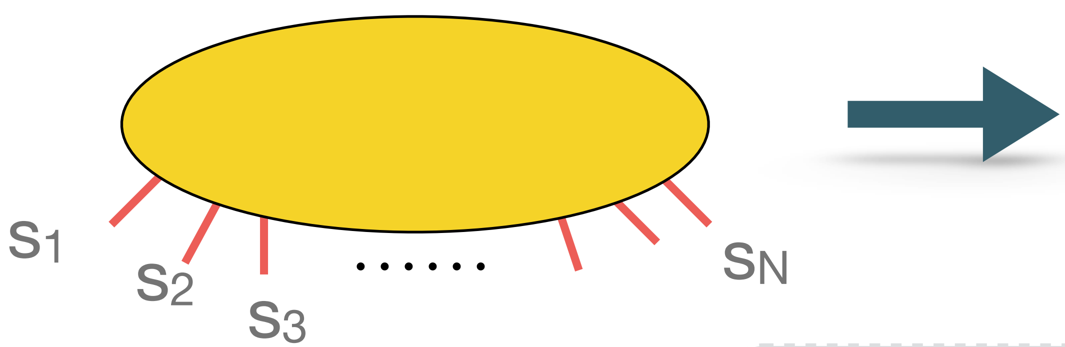

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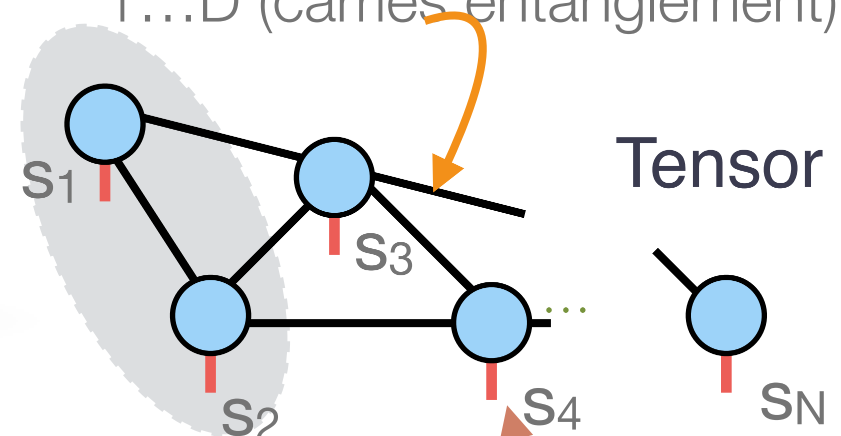


Tensor Network

physical index 1,…,d (local basis)

$$C^{s_1, s_2}_{\alpha\gamma\lambda} = \sum_{\beta} A^{s_1}_{\alpha\beta} B^{s_2}_{\beta\gamma\lambda}$$

$$\mathcal{O}(d^N) \to \mathcal{O}(\text{poly}(N, d, D))$$

$$|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_N} tTr(A^{s_1} A^{s_2} \ldots A^{s_N}) |s_1, s_2, \ldots, s_N\rangle$$

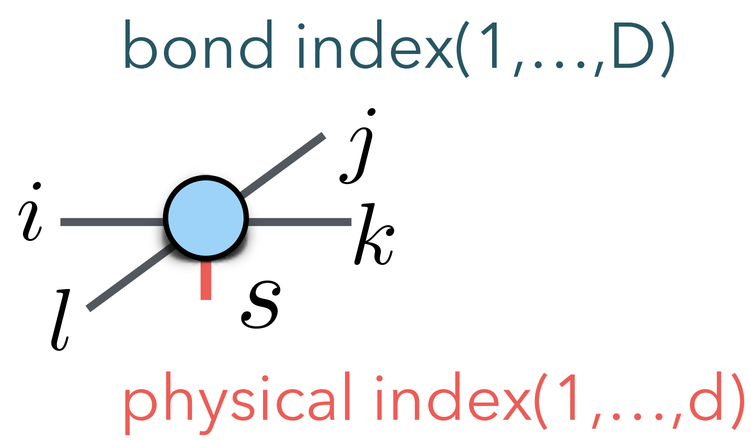
Efficient representation, satisfies area-law, and targets low-energy eigenstates of local Hamiltonians

Tensor network state (TNS)

- Represent wave-function by the tensor network of A tensors

$$|\Psi\rangle = \sum_{s_1,s_2,...,s_N} tTr(A^{s_1}A^{s_2}...A^{s_N})|s_1, s_2, ..., s_N\rangle$$

$$A^s_{ijkl} =$$



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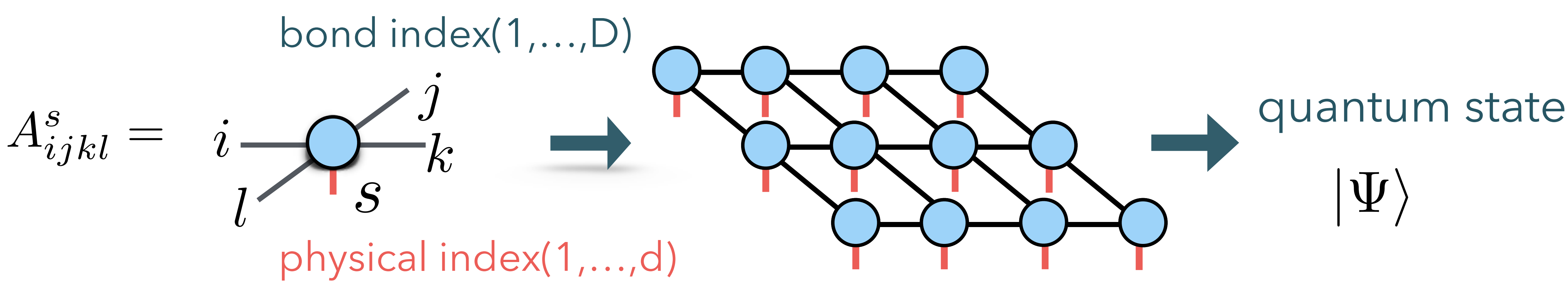


Tensors are local building blocks for
the quantum state (like a DNA, or LEGO)

An example: 1D Affleck-Kennedy-Lieb-Tasaki state state

- The Spin-1 chain [Affleck, I., Kennedy, T., Lieb, E.H., Tasaki '87,88]
- The Hamiltonian of the AKLT point is $H = \sum_i \vec{S}_i \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \vec{S}_{i+1})^2$
- Tensor network states provide a useful numerical tool

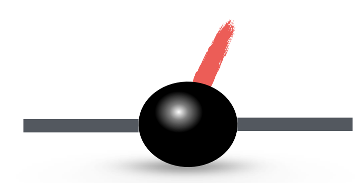
 $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

 $= |+1\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-1\rangle\langle\downarrow\downarrow|$

$$|\Psi\rangle = \sum_{s_1, s_2, ..., s_N} tTr(A^{s_1} A^{s_2} ... A^{s_N})|s_1, s_2, ..., s_N\rangle$$

$$A^{s=+1}_{\uparrow\uparrow} = 1; \quad A^{s=-1}_{\downarrow\downarrow} = 1; \quad A^{s=0}_{\uparrow\downarrow} = A^{s=0}_{\downarrow\uparrow} = \sqrt{\frac{1}{2}}$$

TNS with d=3; D=2

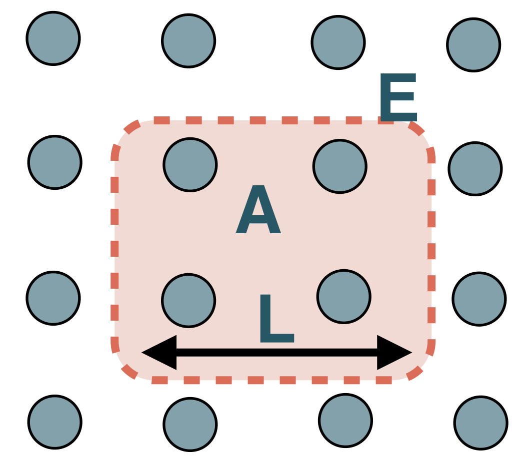


Entanglement

- Entanglement → key resource in quantum information
- 2- dimensional system

Reduced density matrix of subsystem A $\rho_A = tr_E(|\Psi\rangle\langle\Psi|)$

Entanglement entropy $S(A) = -tr(\rho_A \log \rho_A)$



For many ground states → $S(A) \sim L \qquad (L > \xi)$

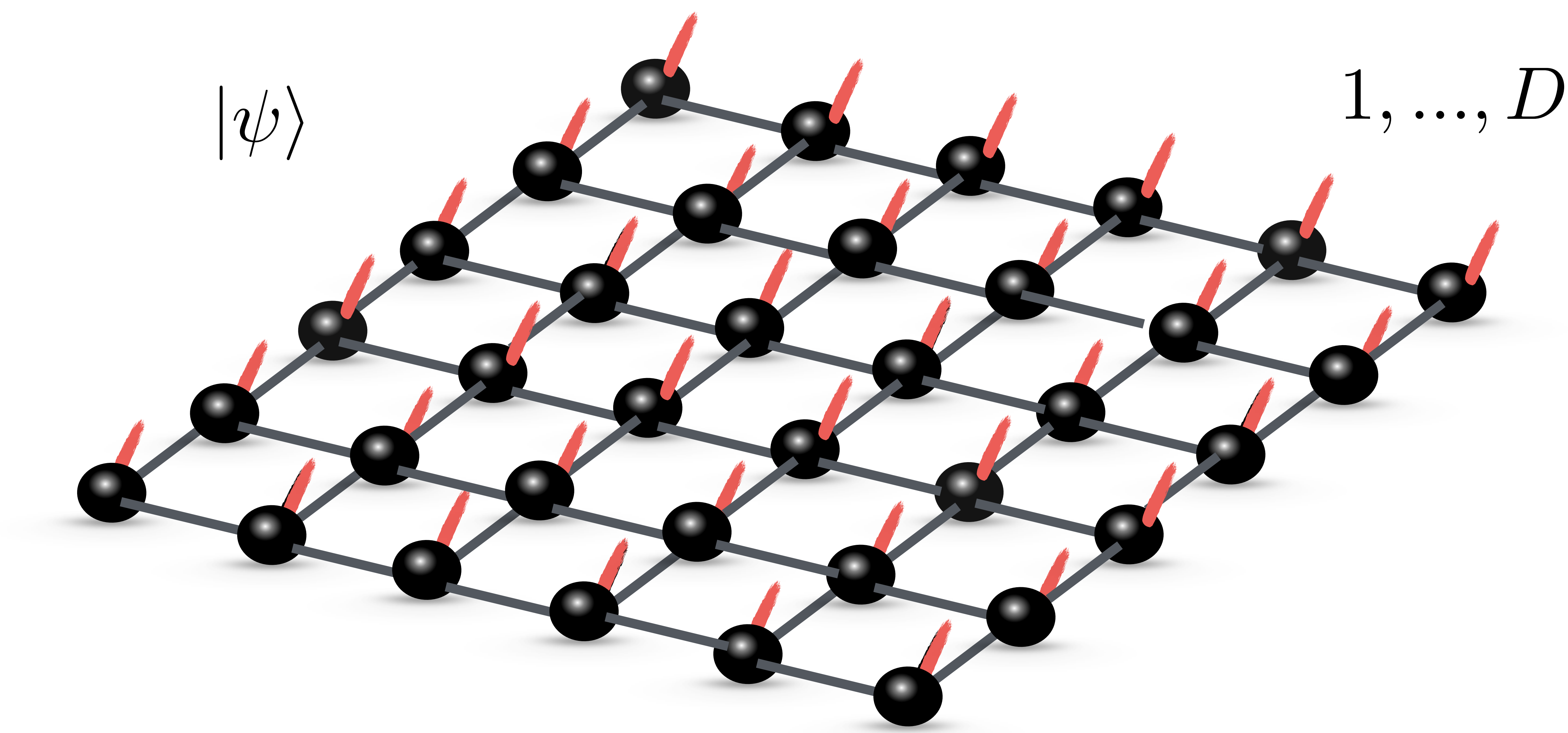
- In d dimensional system

Generic state $S(A) \sim L^d$

Ground state of local Hamiltonian $S(A) \sim L^{d-1}$

[Srednicki, Plenio, Eisert, Dreißig, Cramer, Wolf…]

TNS obey area law



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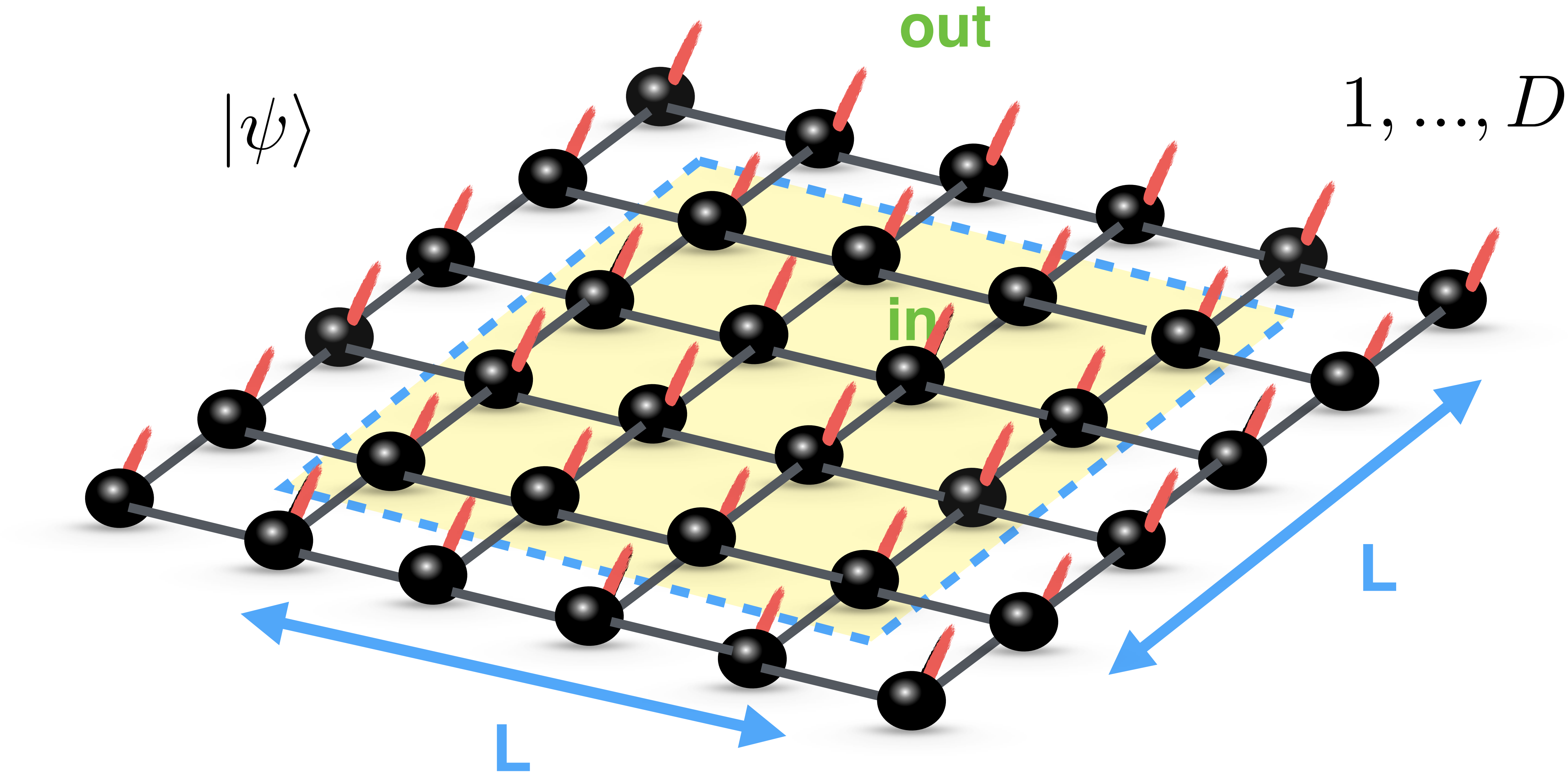
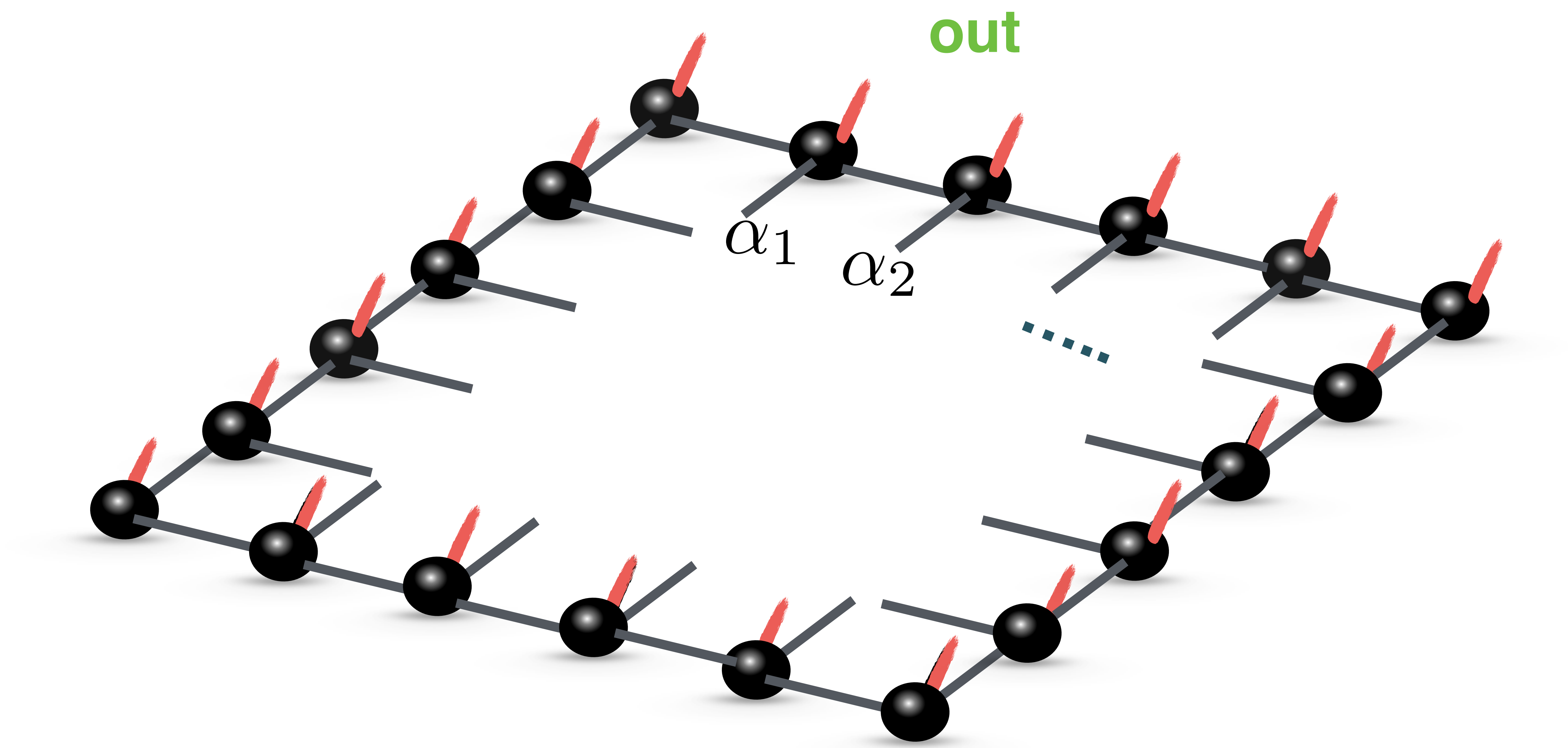
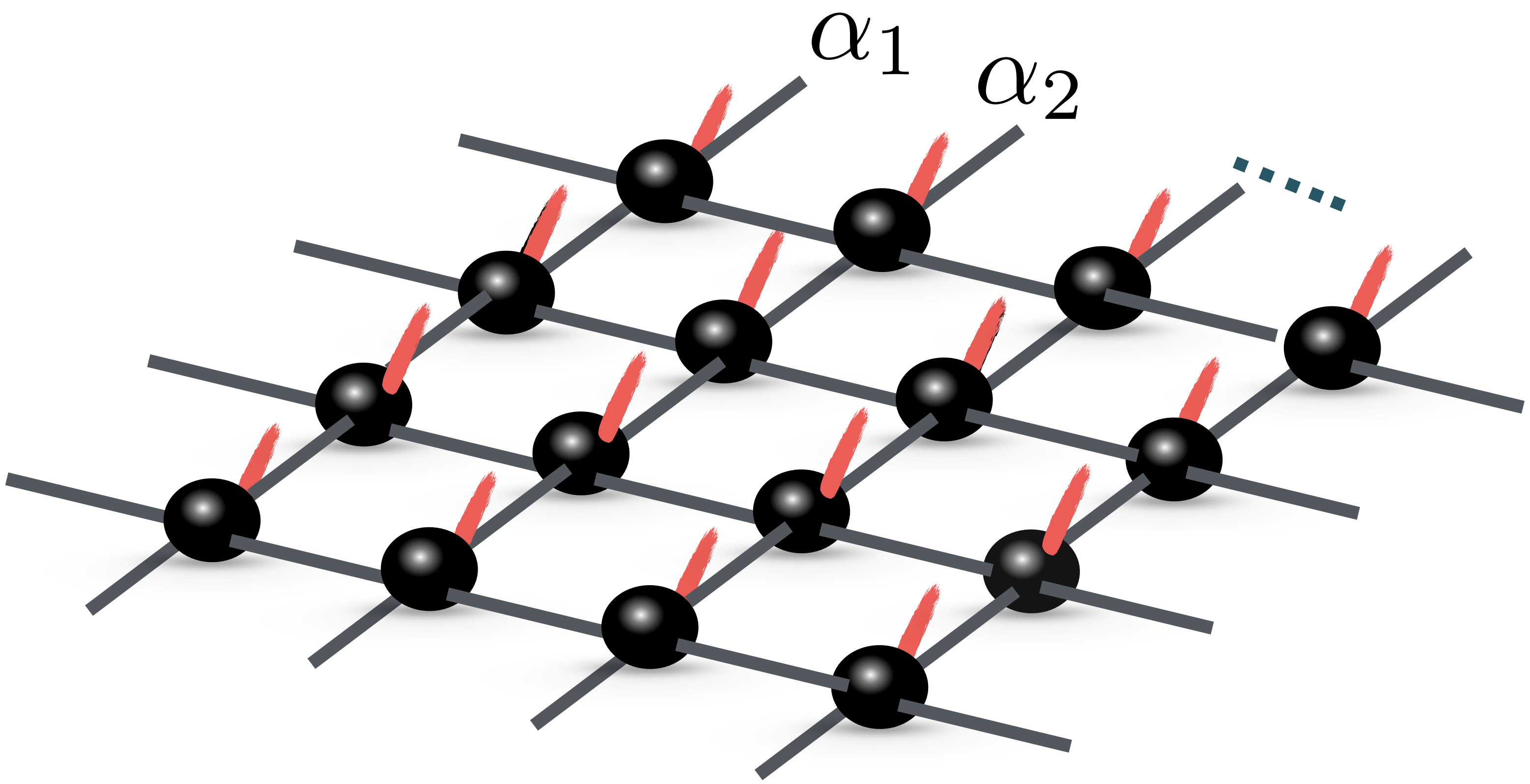




$|out(\bar{\alpha})\rangle \qquad \bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_{4L-1}, \alpha_{4L})$

$\bar{\alpha} = 1, 2, ..., D^{4L}$

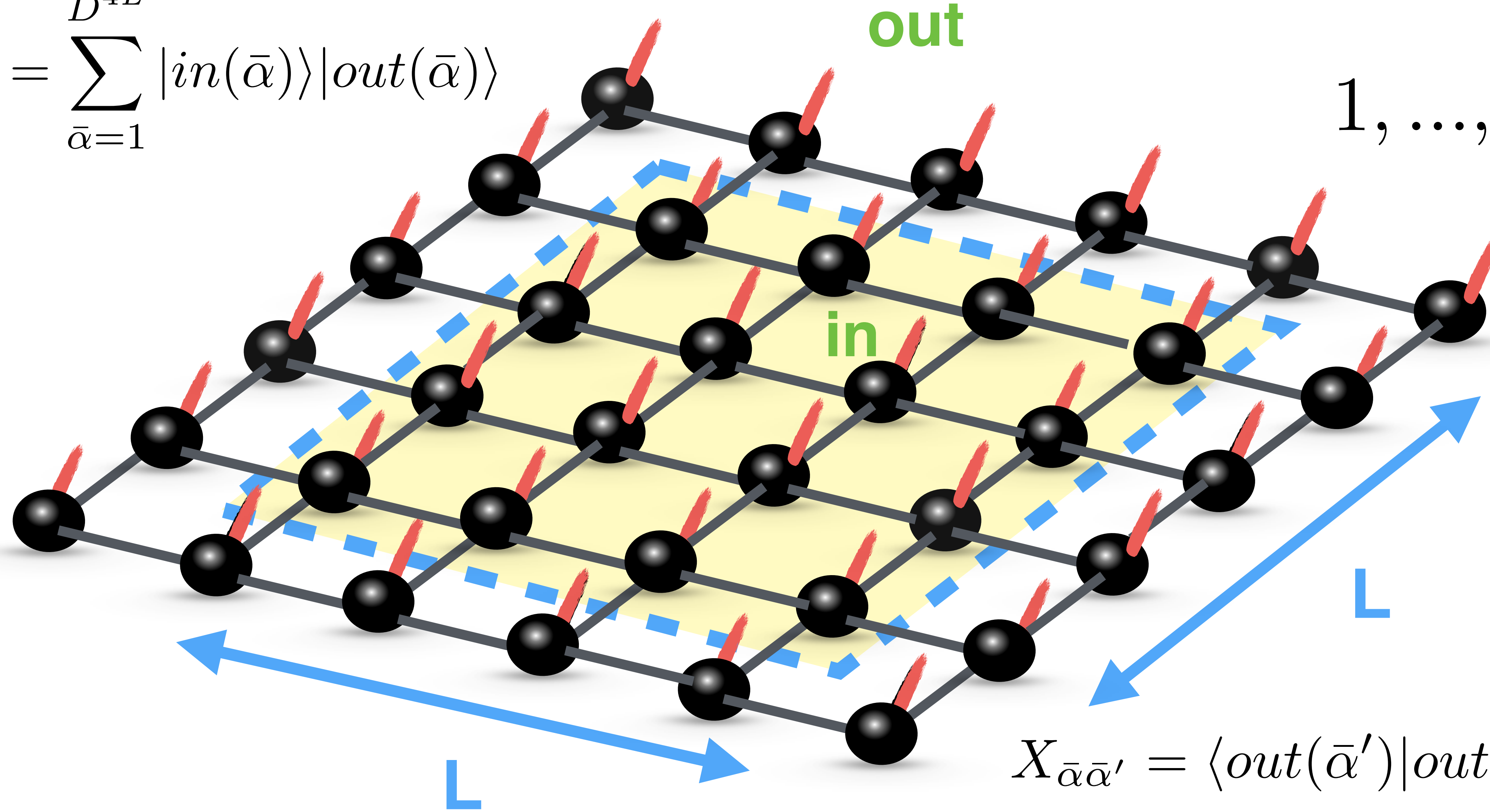
in



$|in(\bar{\alpha})\rangle \qquad \bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_{4L-1}, \alpha_{4L})$

$\bar{\alpha} = 1, 2, ..., D^{4L}$

$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$



$$1, \ldots, D$$

$$X_{\bar{\alpha}\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

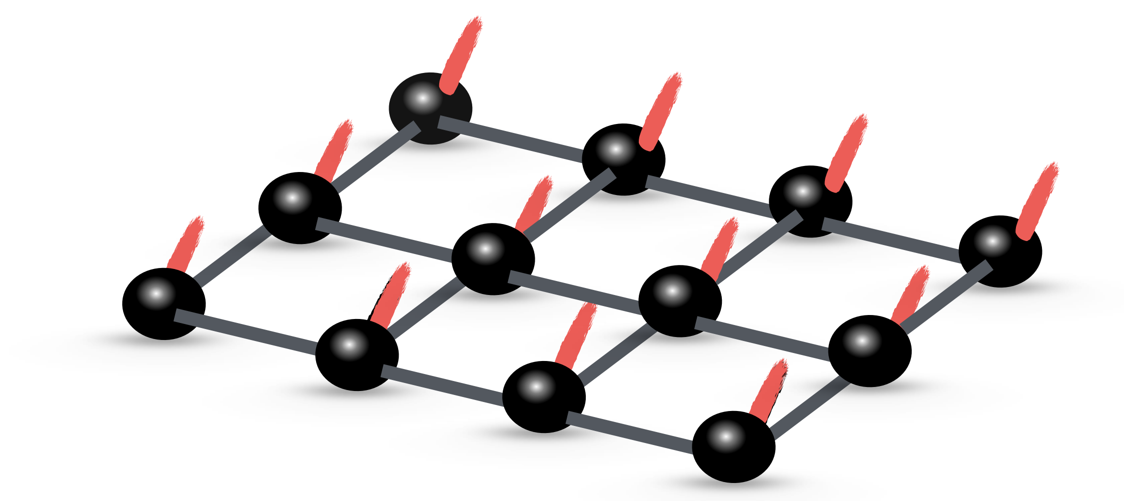
$$\rho_{in} = tr_{out}(|\Psi\rangle\langle\Psi|) = \sum_{\bar{\alpha}\bar{\alpha}'} X_{\bar{\alpha}\bar{\alpha}'} |in(\bar{\alpha})\rangle\langle in(\bar{\alpha}')|$$

$$rank(\rho_{in}) \leq D^{4L} \qquad S(L) = -tr(\rho_{in} \log \rho_{in}) \leq \log(D) 4L$$

size of the boundary

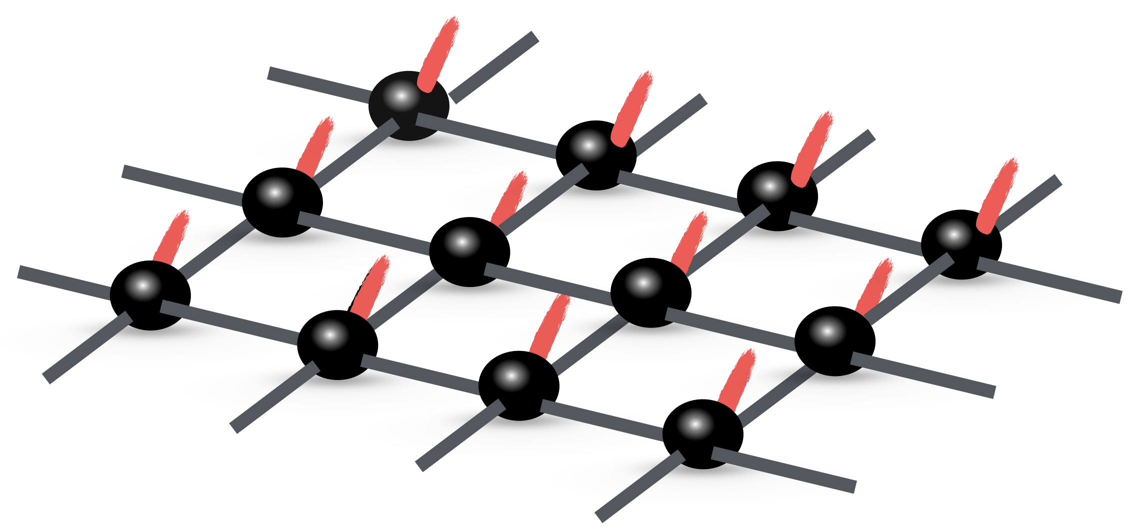
Infinite system

finite TNS



[F. Verstraete, I. Cirac 06']

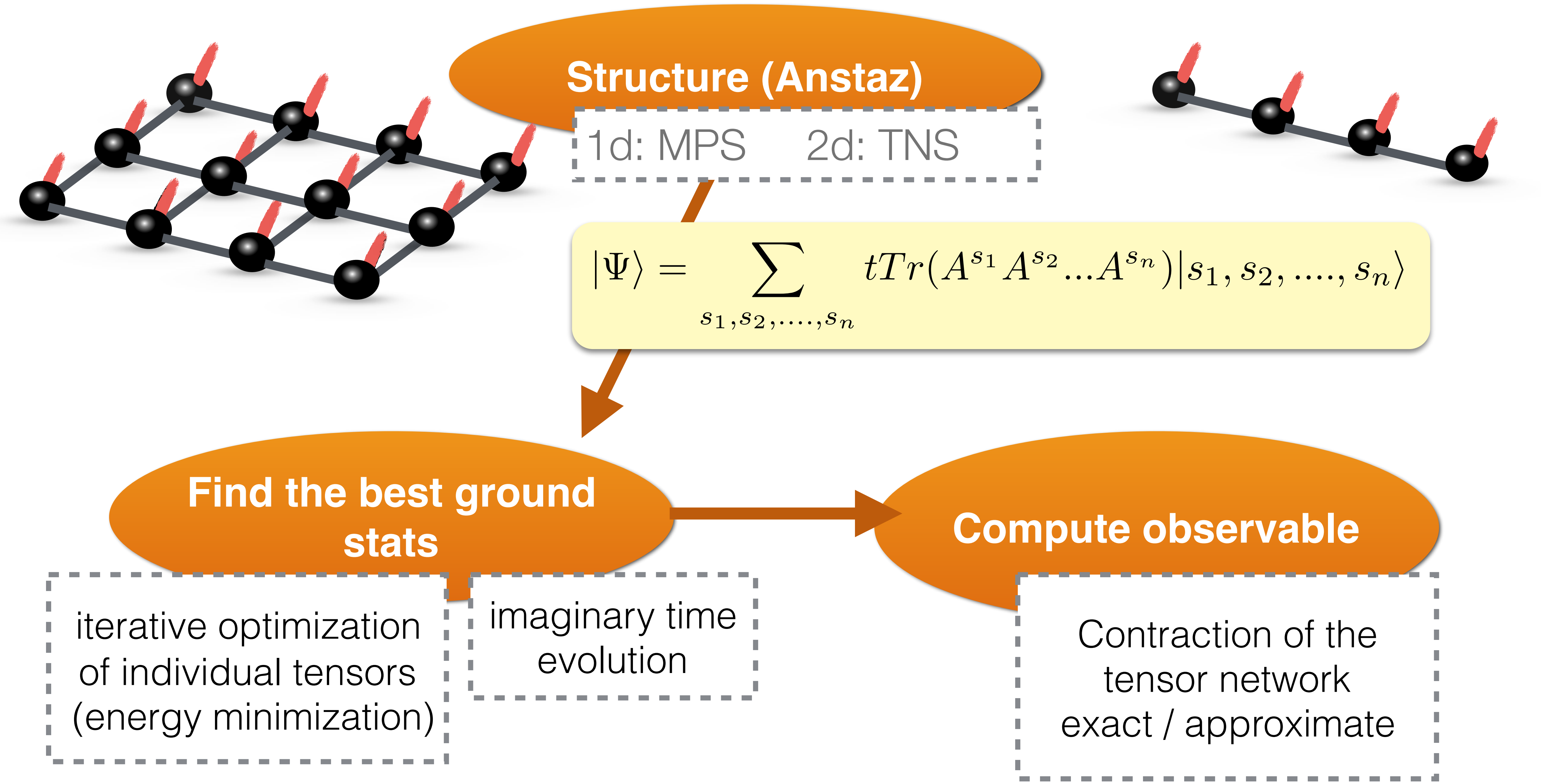
infinite TNS



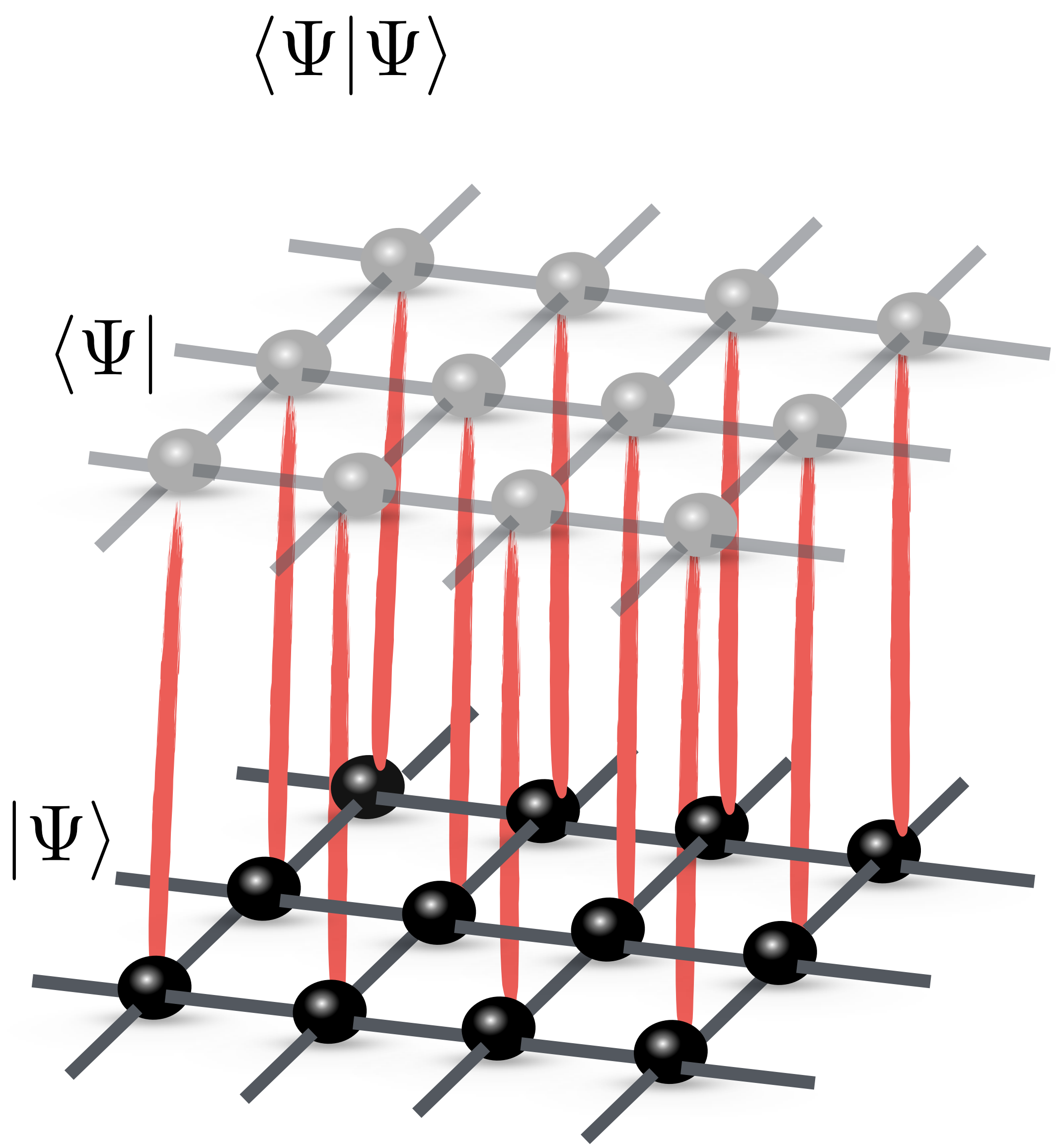
Unit cell of tensors is repeated periodically over the whole PEPS: translational invariance

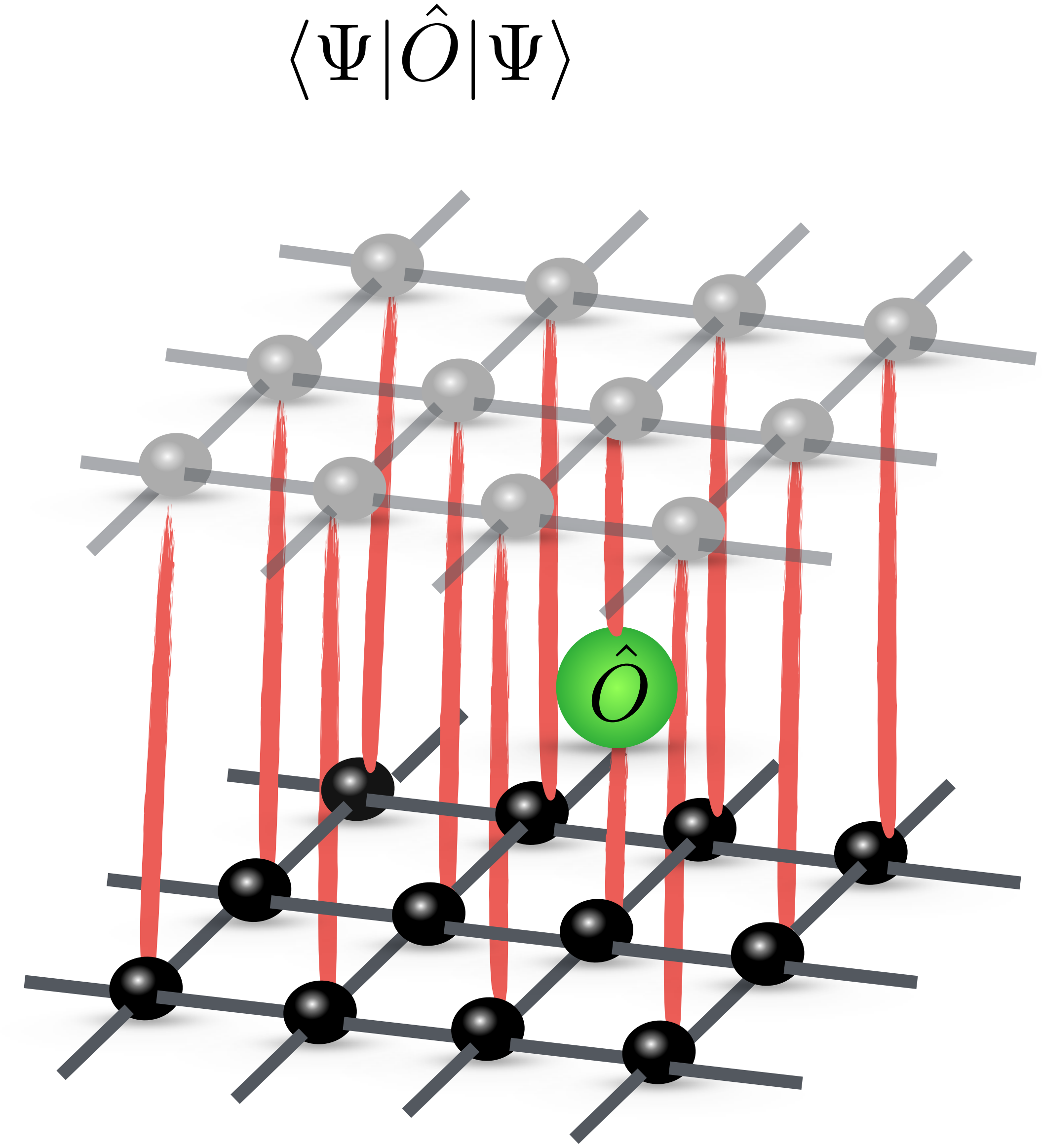
[J. Jordan, R. Orus, G. Vidal, F. Verstraete, I. Cirac, 08']

Tensor network algorithm

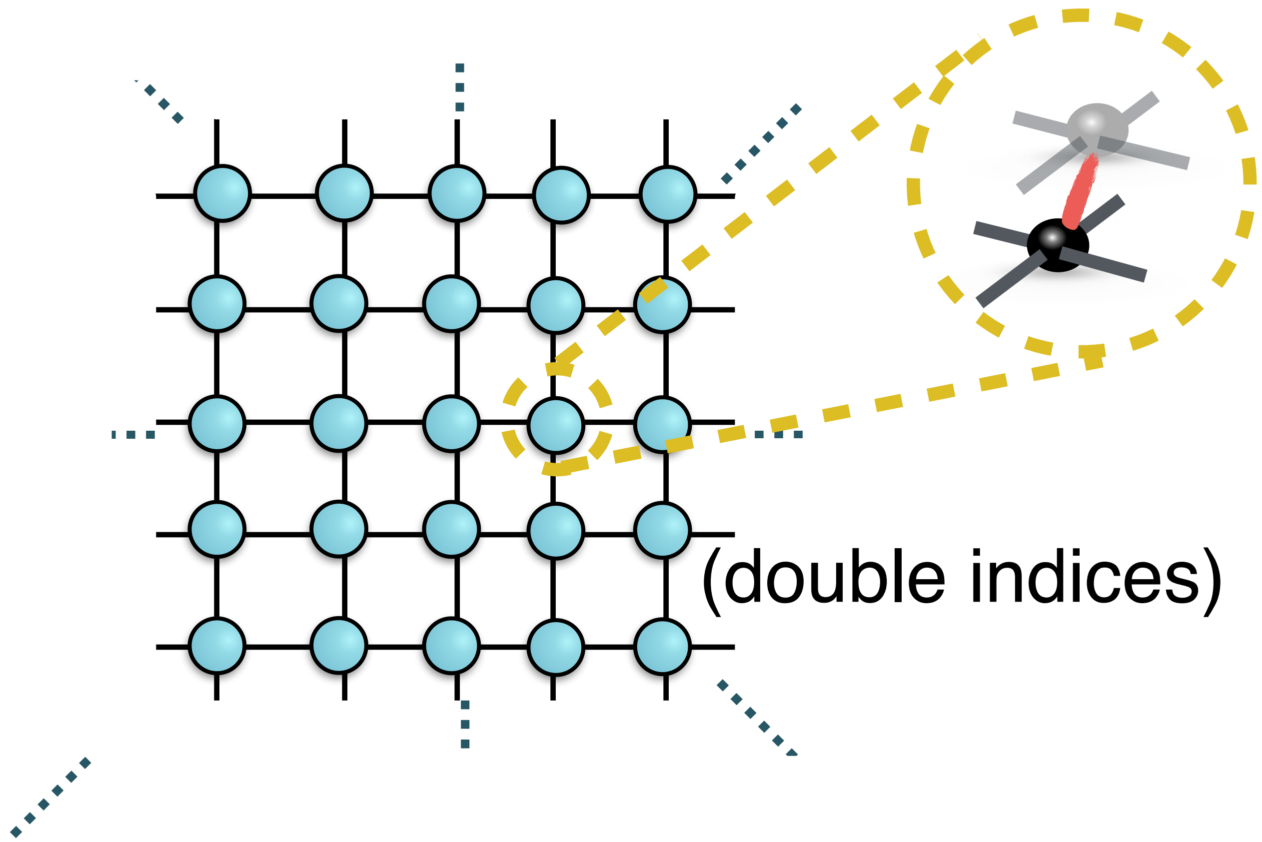


Determine the observables





Contracting the infinite 2d lattice

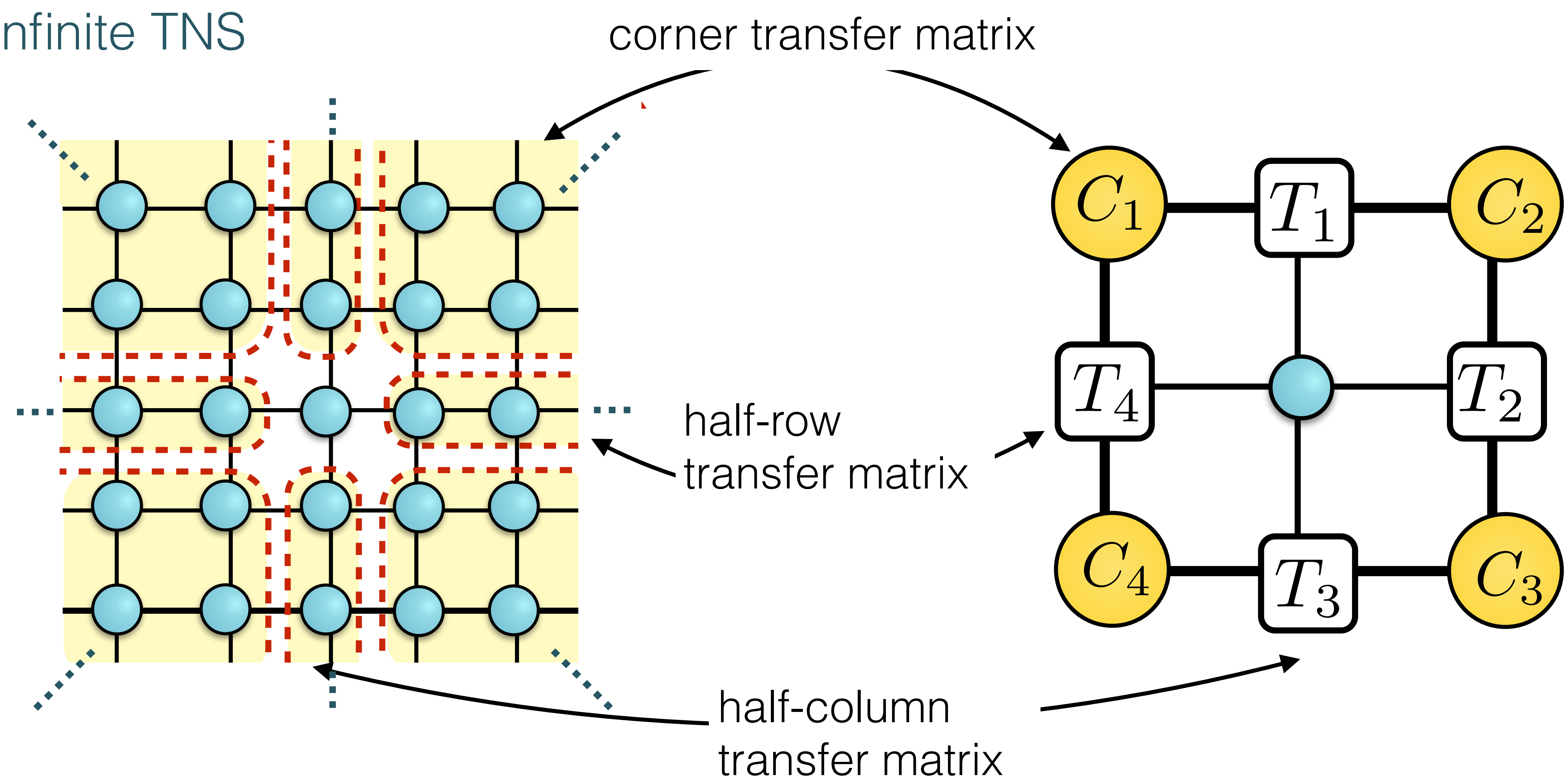


To determine observables

Contraction of this infinite lattice

Contracting the infinite 2d lattice

from infinite TNS



Renormalized Corner Transfer Matrices

CTM method

[R. Orús, G. Vidal,09', R. Orús,12']

CTM method

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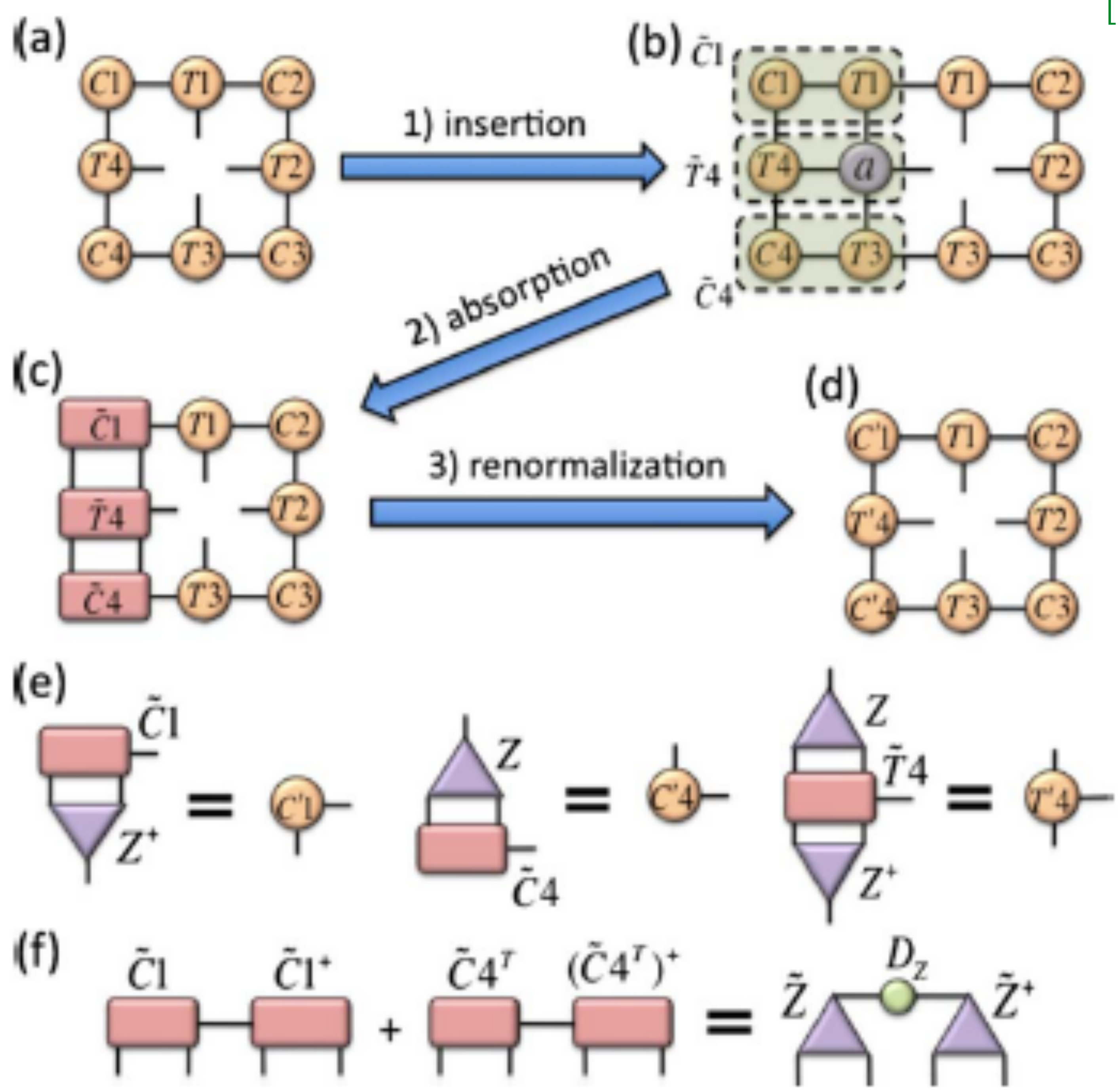

Outline

- Introduction
  - Tensor network state
  - corner tensor
  - Symmetry protected topological order phases

- ***The fingerprints of universal physics are encoded holographically in numerical CTMs and CTs.***
  - classical and quantum Ising (quantum-classical correspondence)
  - deformed symmetry protected topological order phase
  - 2d quantum XXZ model

- Quantum state renormalization in 2D using corner tensors
  - chiral topological PEPS

- Summary

Corner tensor

Corner tensor

- Corner transfer matrices (CTMs) method can be used to study physical system.

[R. J. Baxte 1968; T. Nishino and K. Okunishi 1996; R. Orús 2012]

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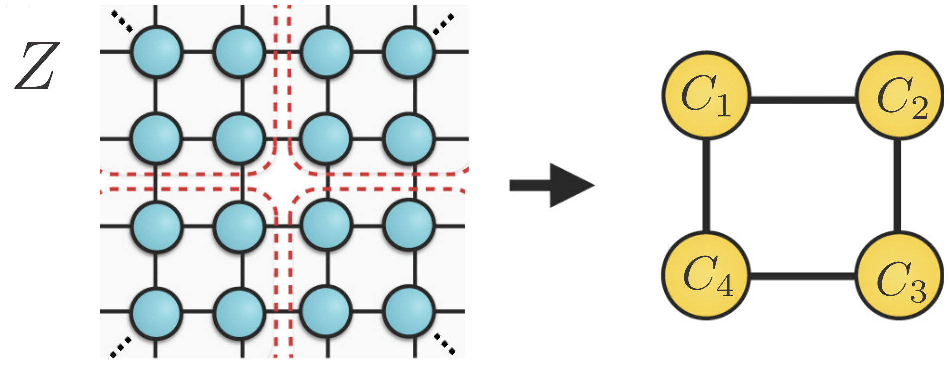
✓ The Partition function of classical lattice model

$$Z = \mathrm{tr}(C_1 C_2 C_3 C_4),$$



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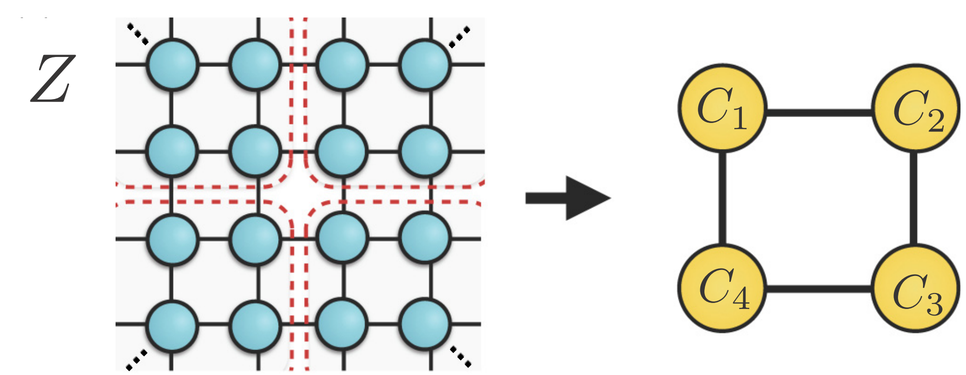

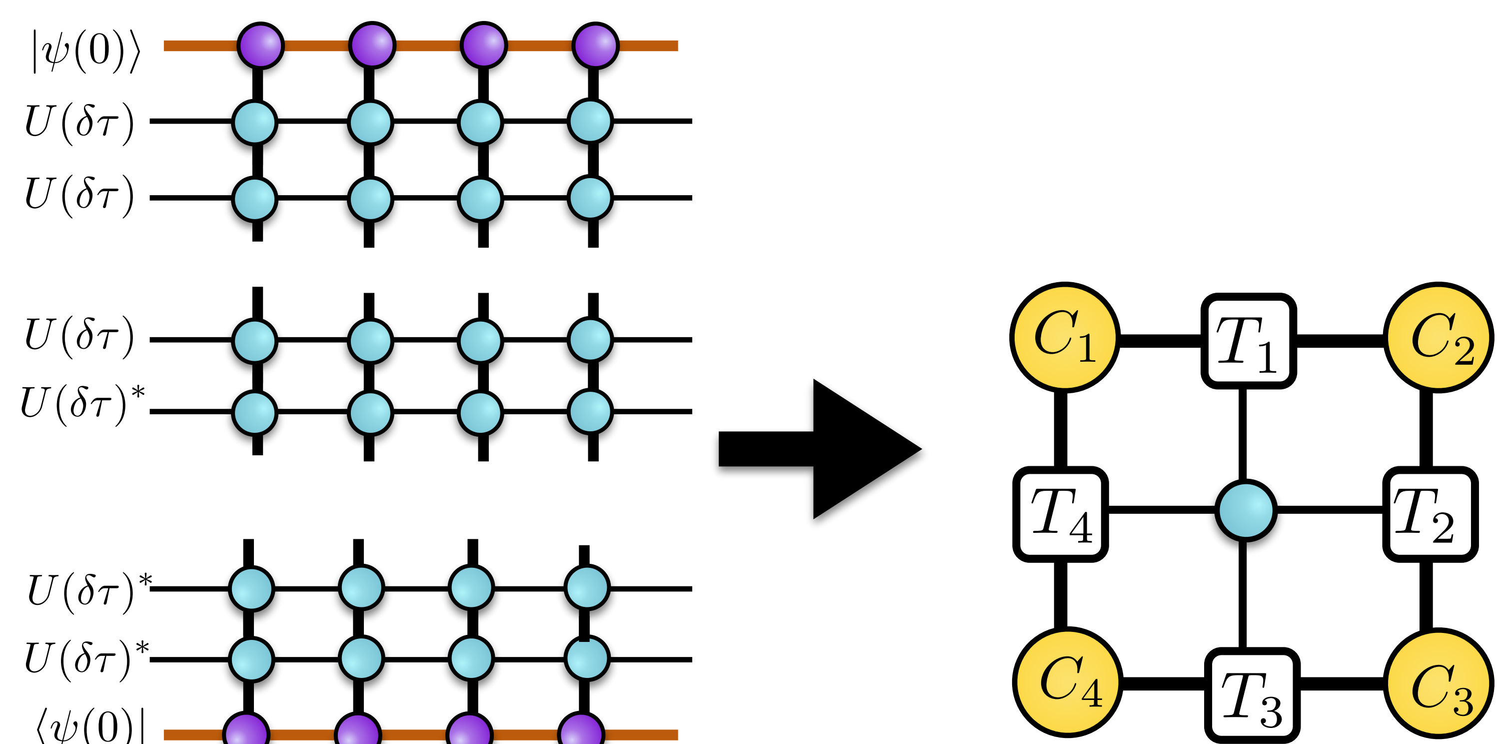

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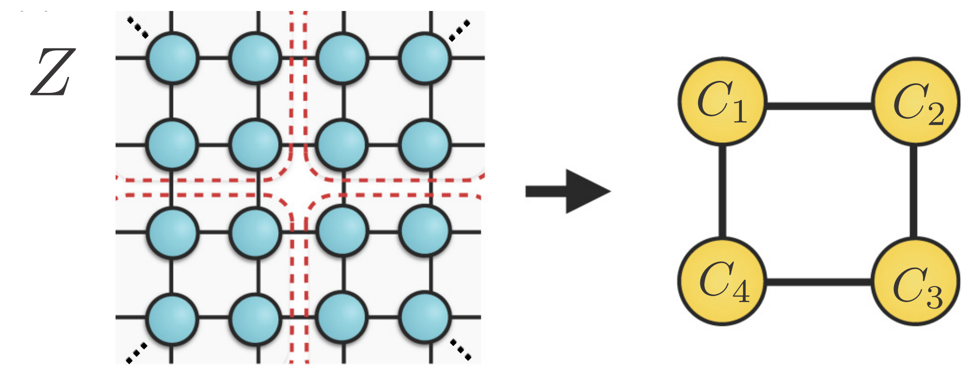
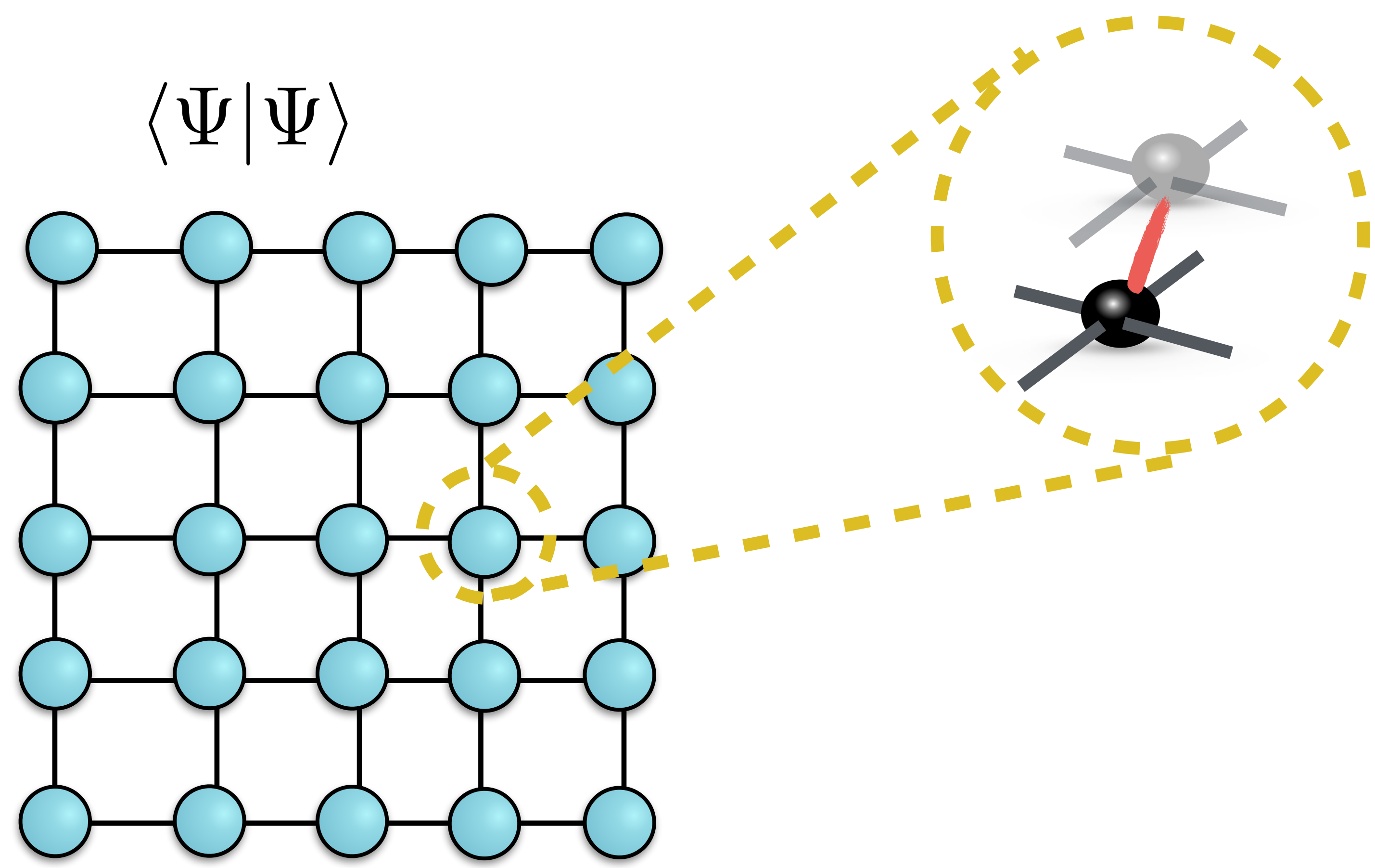


$$\langle \Psi | \Psi \rangle$$



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1d quantum Ising model

$$H_q = -\sum_{i=1}^{L-1} \sigma_x^{[i]} - \delta \sigma_x^{[L]} - \lambda \sum_{i=1}^{L-1} \sigma_z^{[i]} \sigma_z^{[i+1]},$$

$$[H_q, H_c] = 0$$

↔

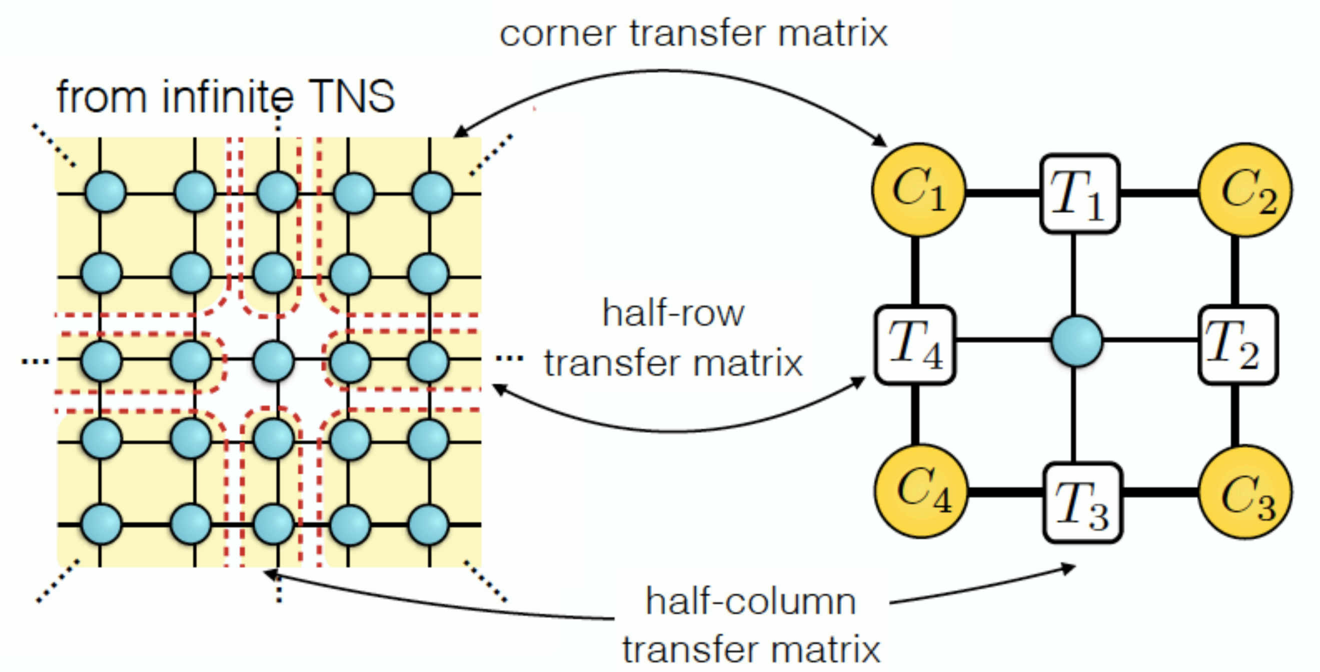
2d classical Ising model Hc with an isotropic coupling K

$$\delta = \cosh 2K \quad \text{and} \quad \lambda = \sinh^2 K,$$

Corner tensor

Corner tensor

- Start from a Hamiltonian or a wave function
  - form a tensor network
  - use CTM method to study the properties of ground state



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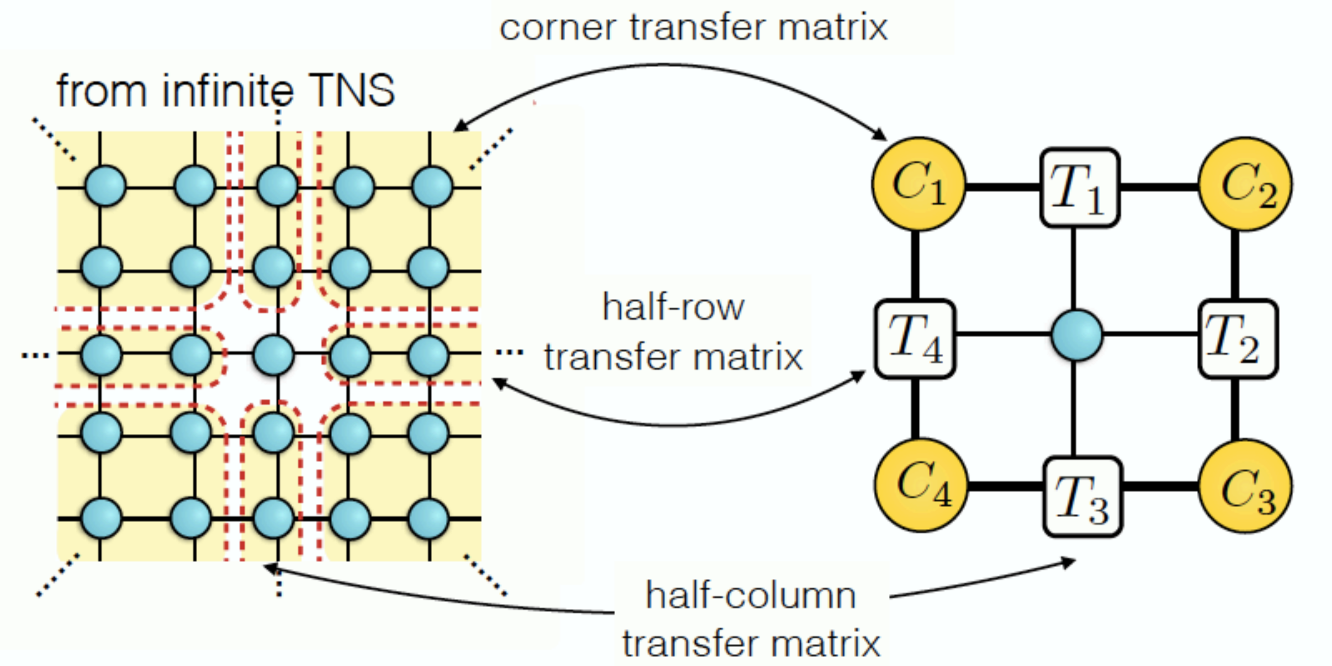

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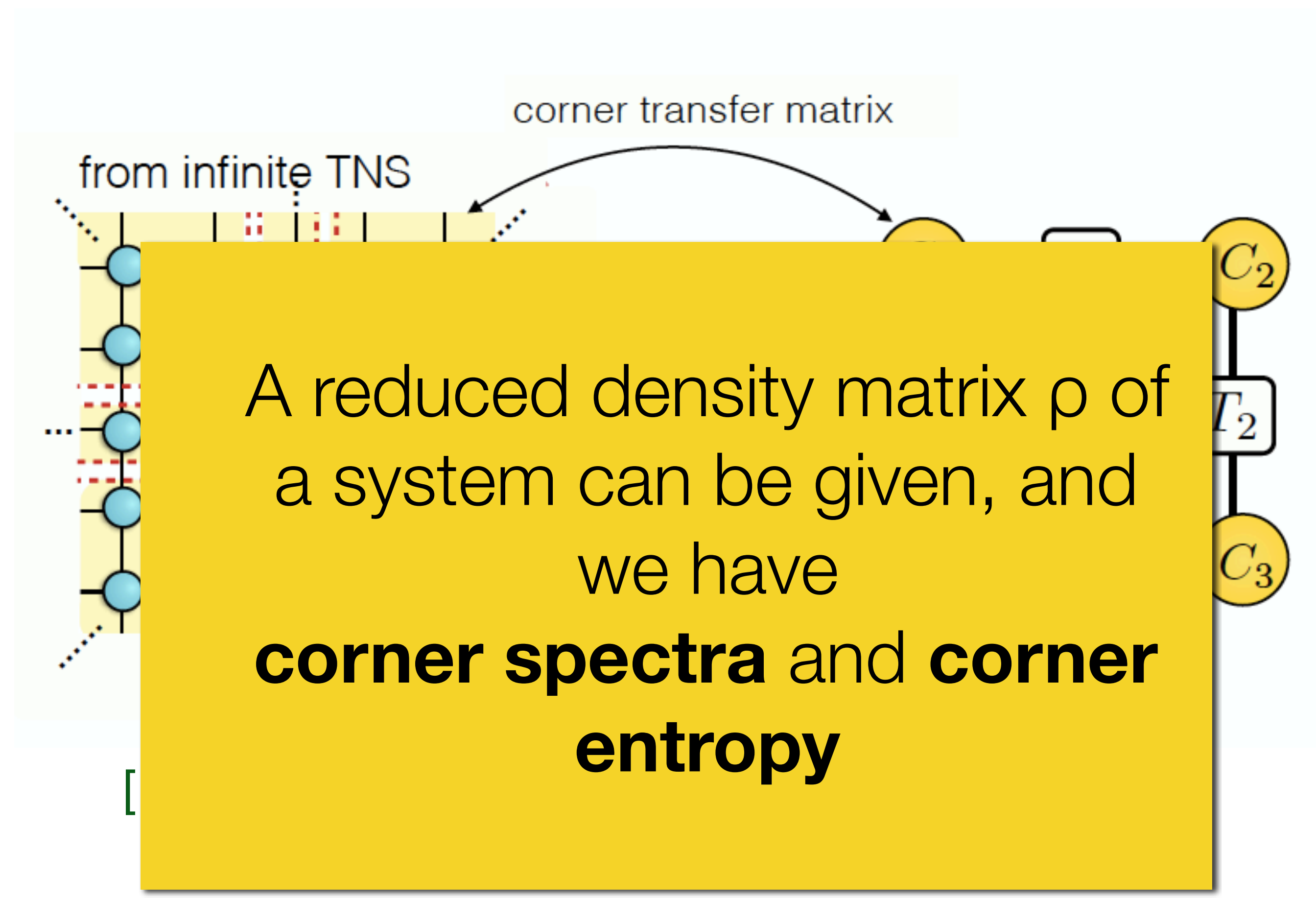
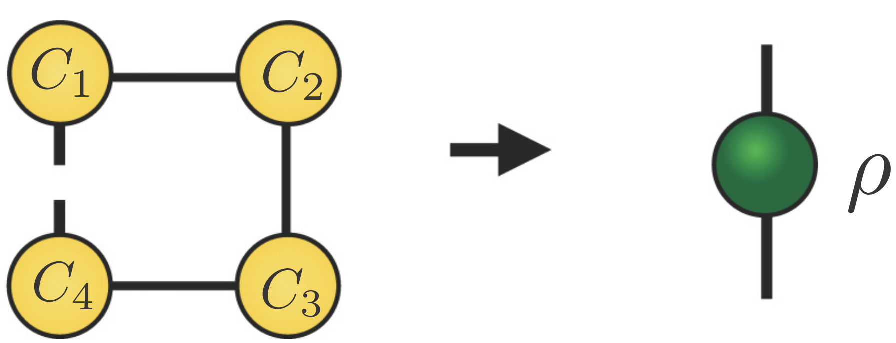
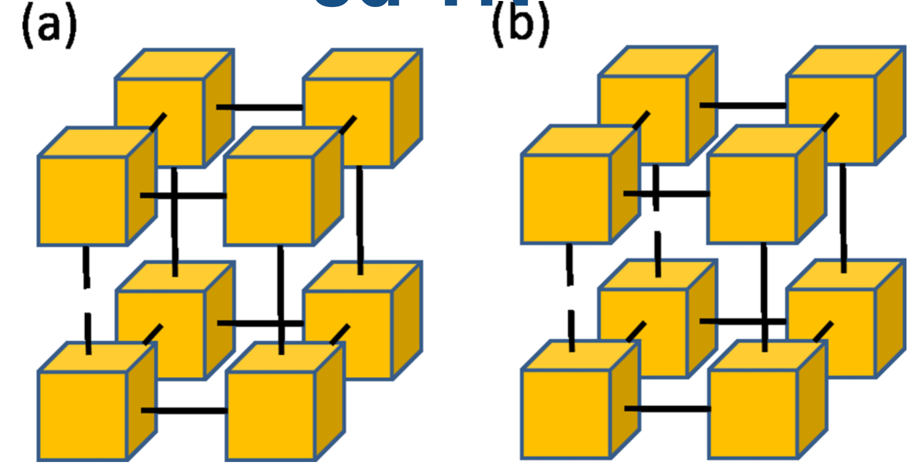


A reduced density matrix ρ of a system can be given, and we have **corner spectra** and **corner entropy**

2d TN

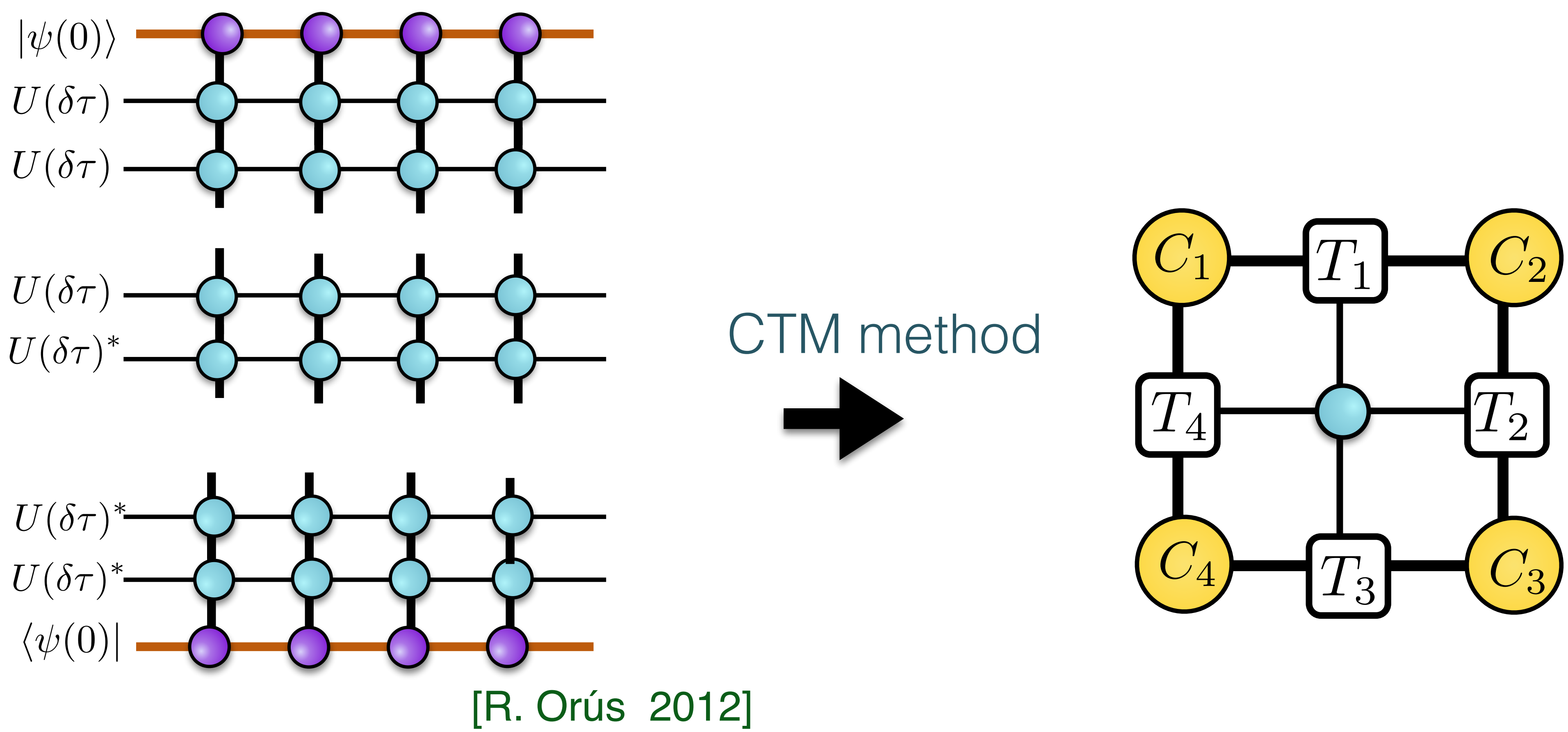


3d TN



CTM from tensor network (TN)

- For 1d quantum: from Hamiltonian ➜ (1+1)d TN



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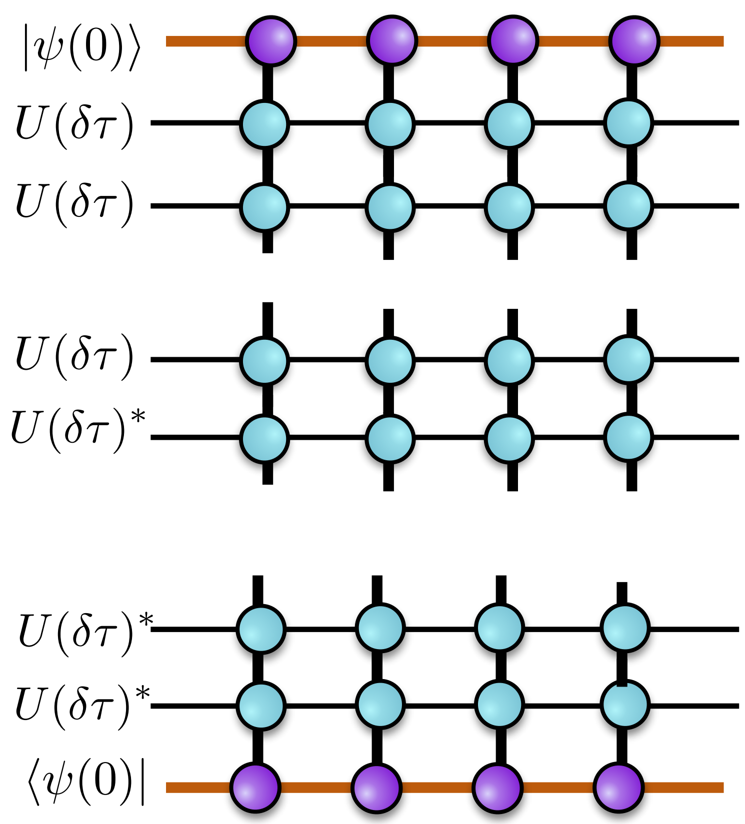
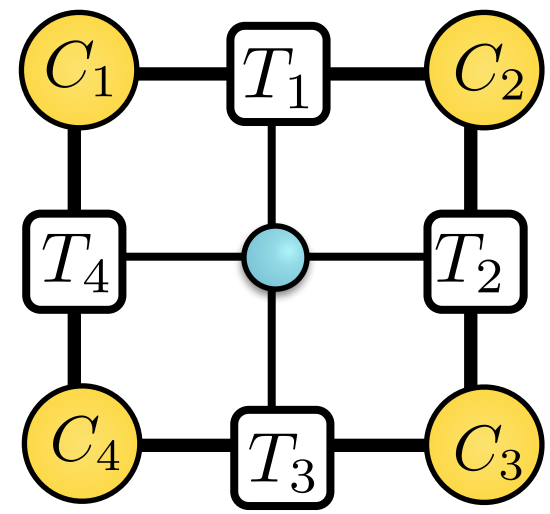
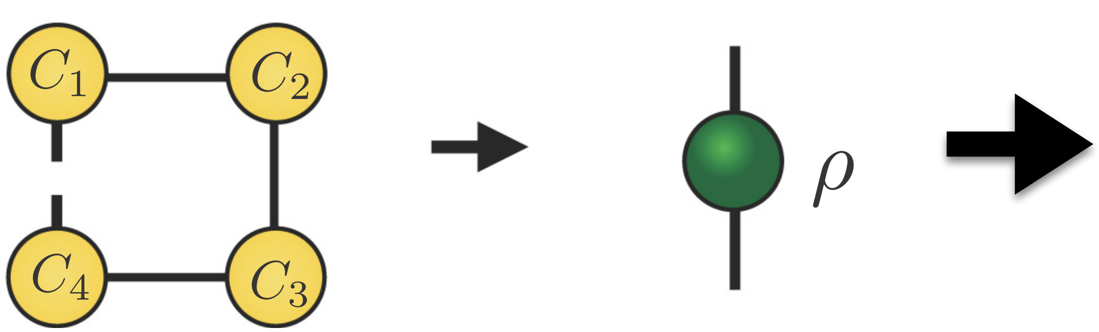


[R. Orús 2012]

CTM method





corner spectrum

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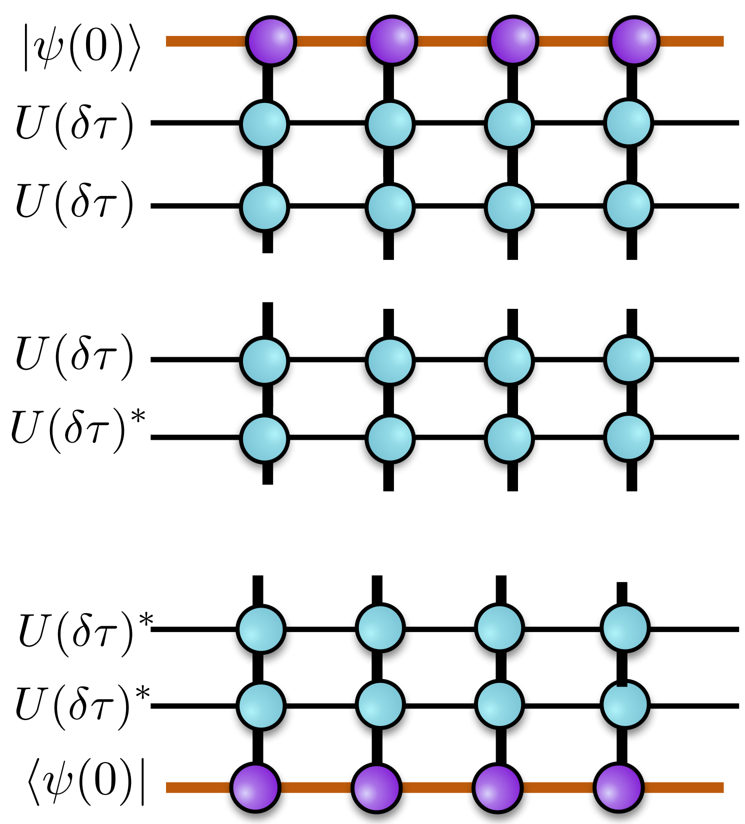

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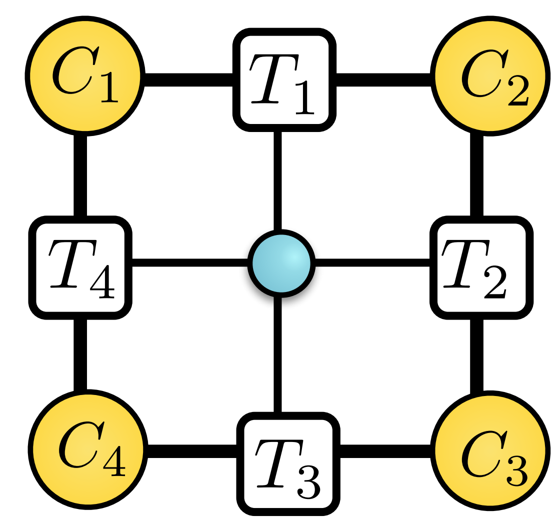

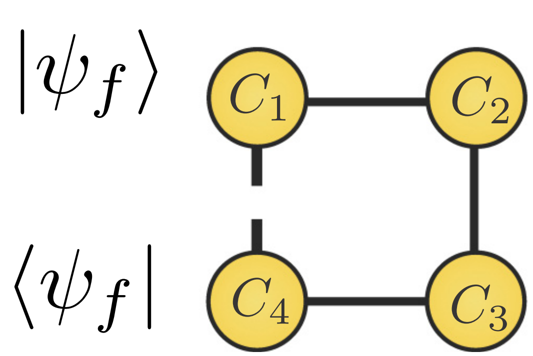

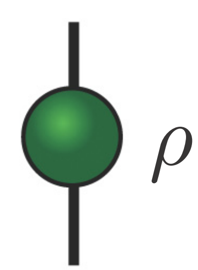

corner spectrum

||

entanglement spectrum

CTM from tensor network (TN)

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CTM from tensor network (TN)

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  (a) use the 2d quantum state renormalization ➜ 3d TN

  (b) the norm of PEPS ➜ 2d TN

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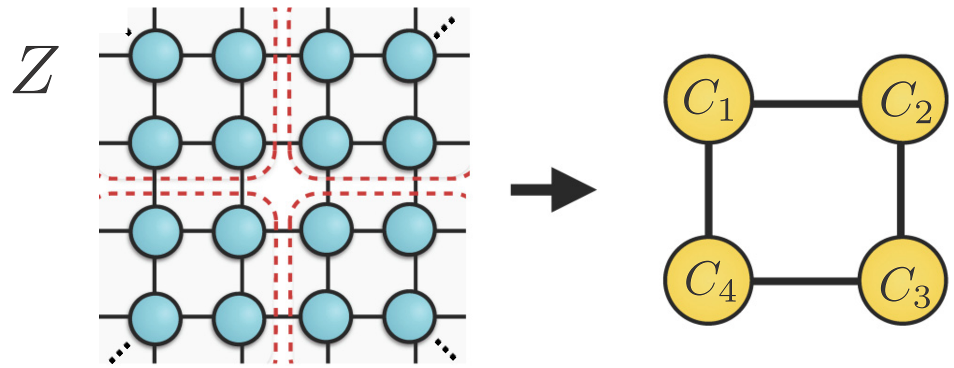

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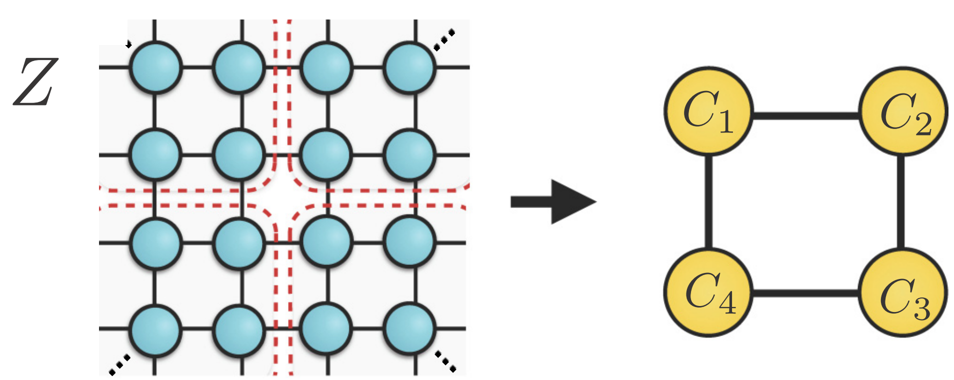

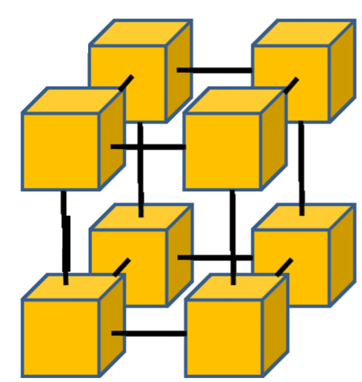

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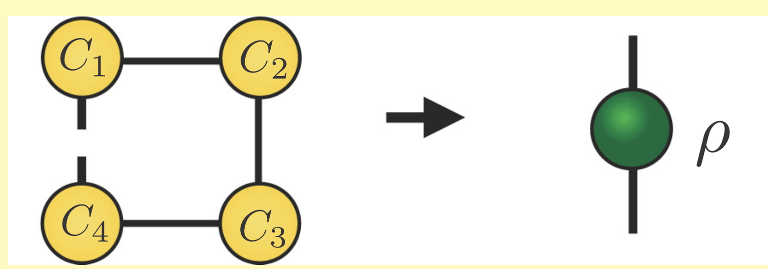

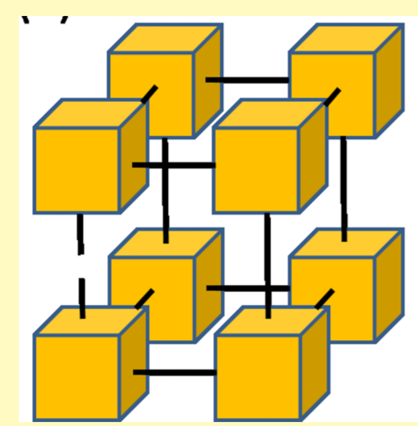

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corner spectrum

=

entanglement spectrum

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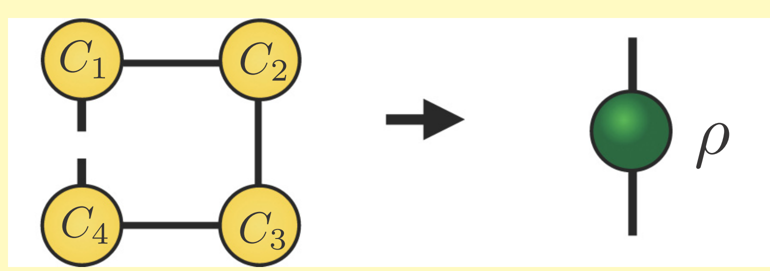
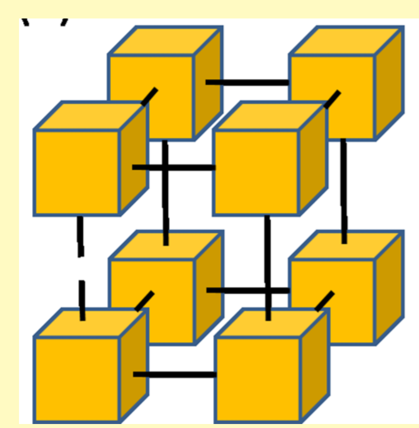
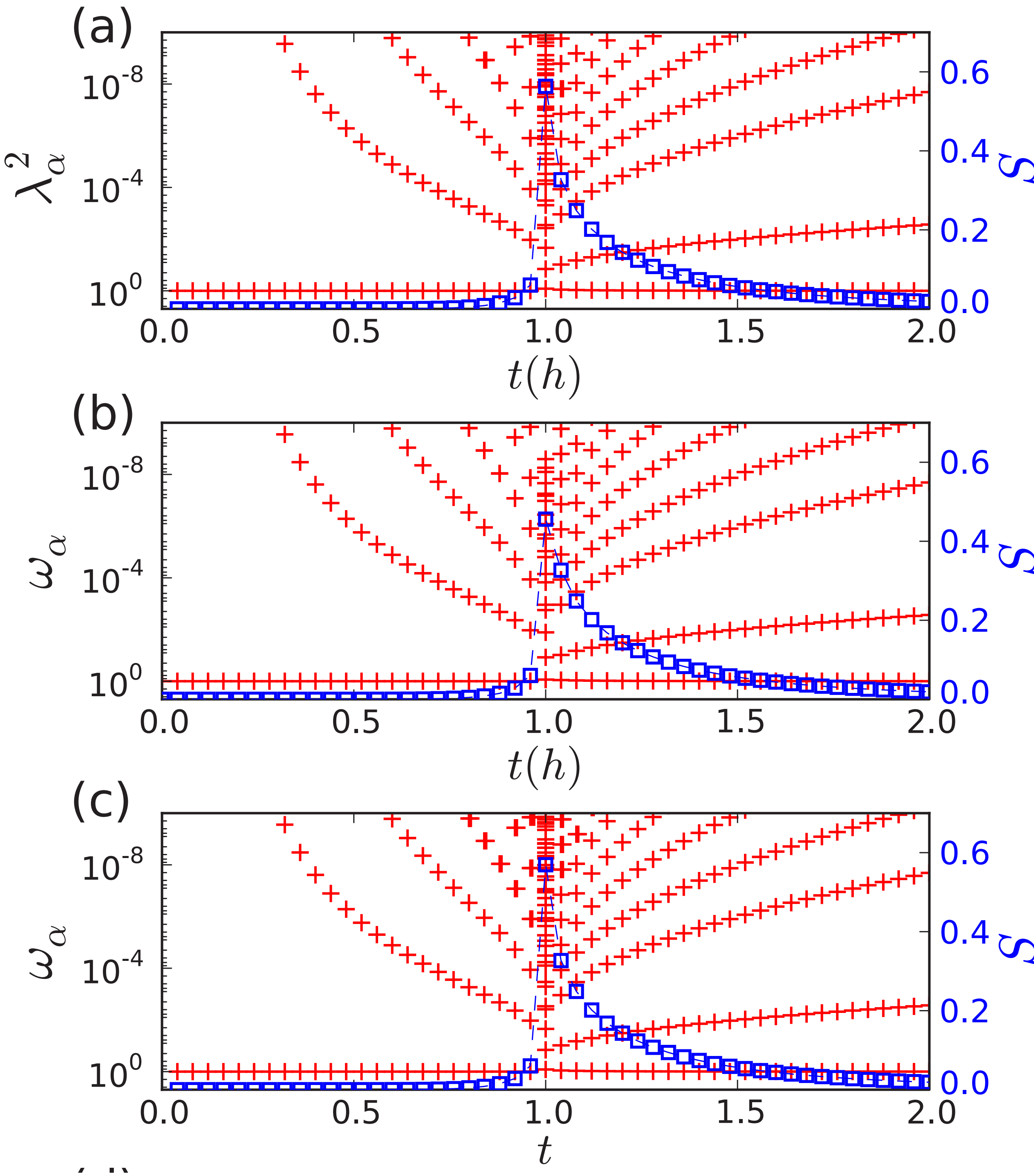




1d quantum Ising universality class

- 1d quantum Ising model:

$$H_q = -\sum_i \sigma_x^{[i]} \sigma_x^{[i+1]} - h \sum_i \sigma_z^{[i]},$$



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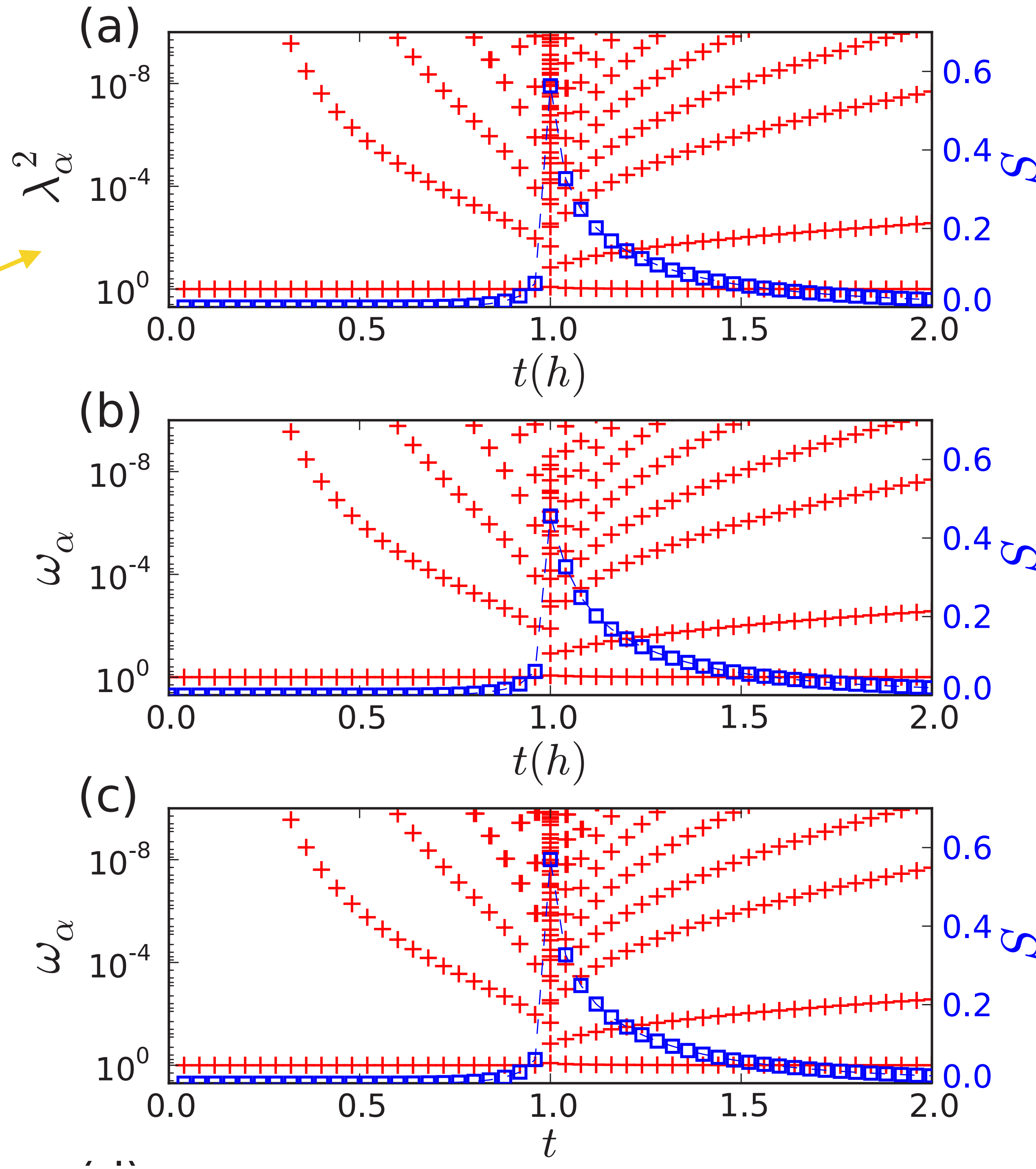

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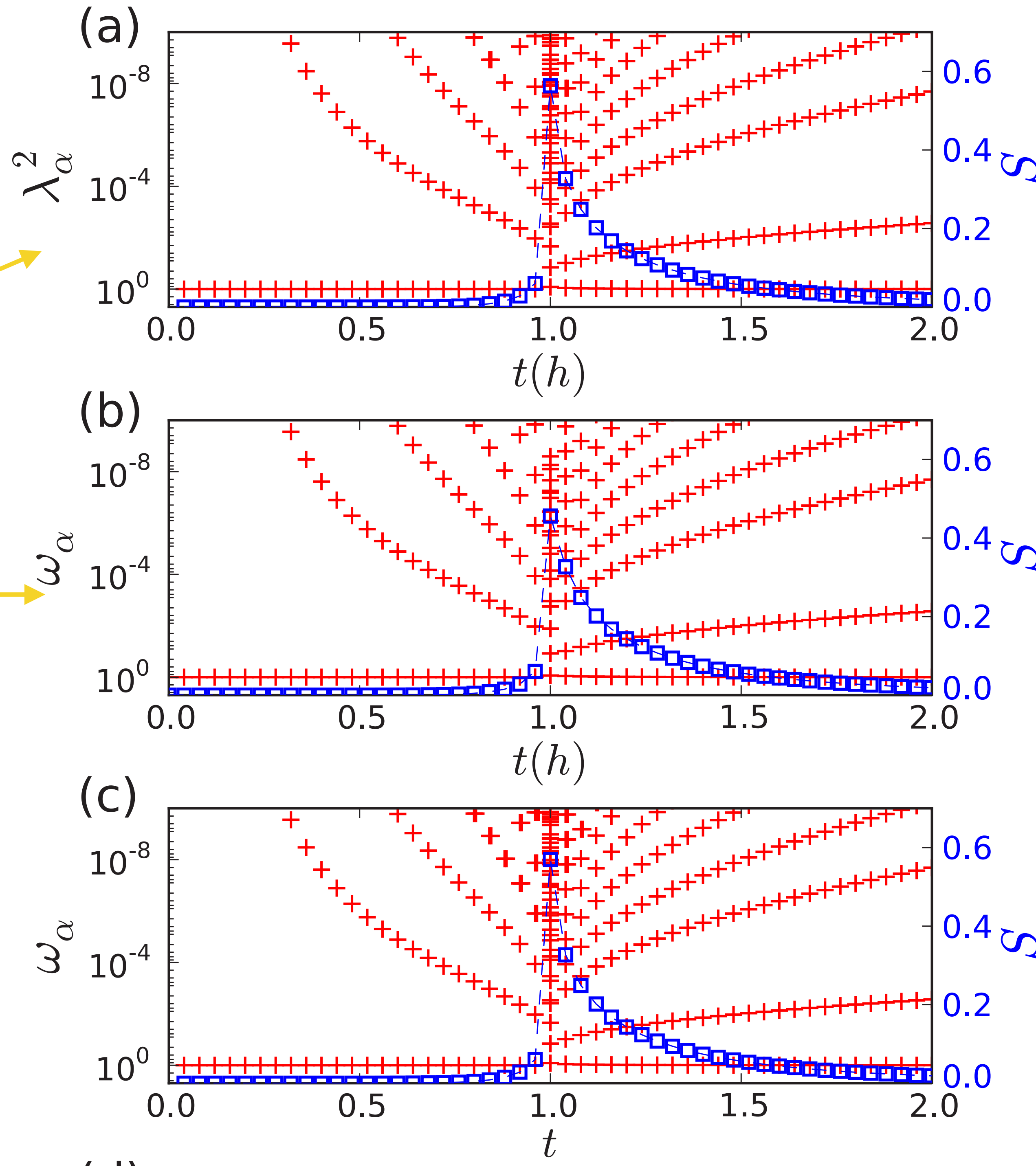

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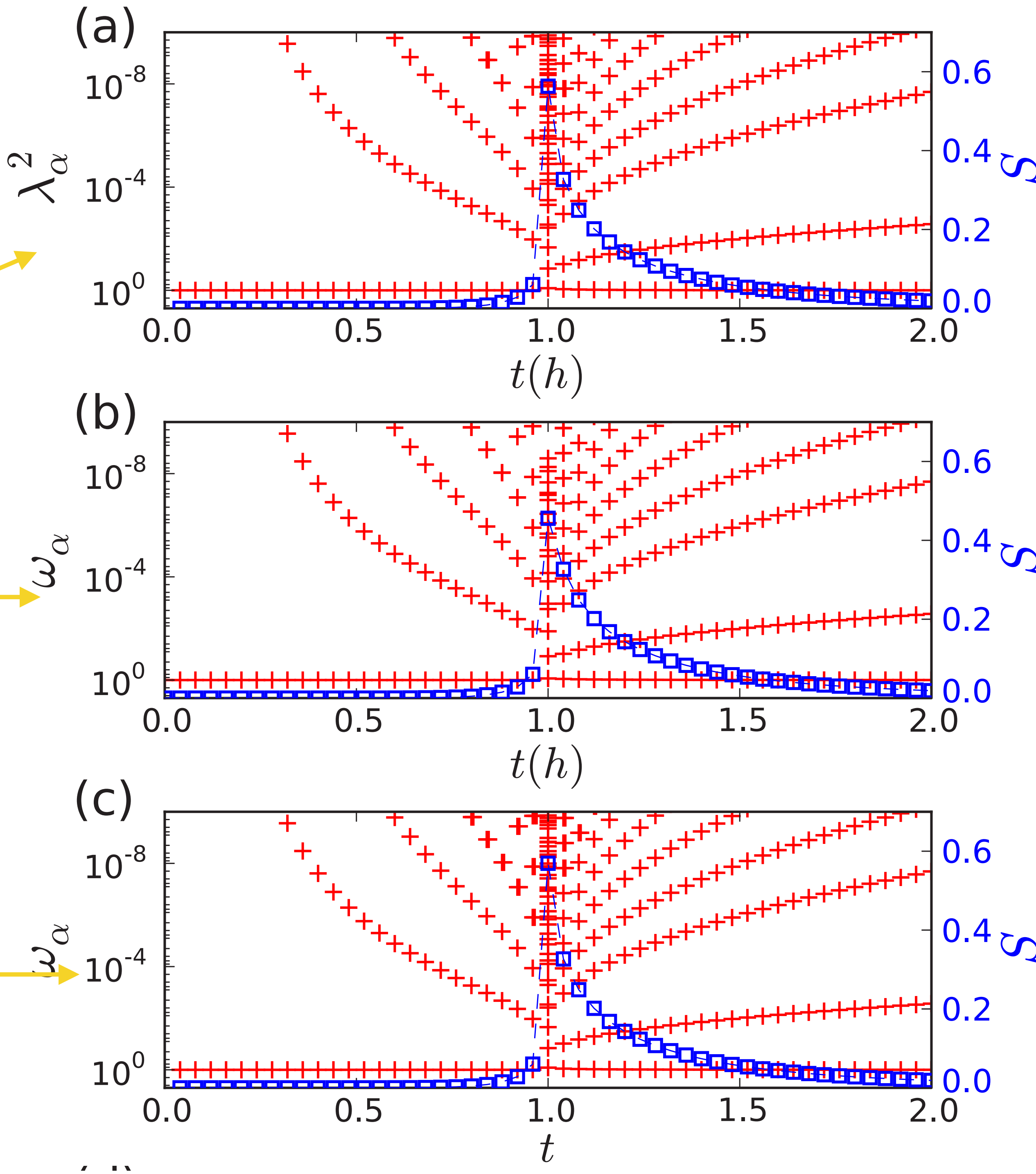
- 2d classical model

(c) To compute the corner spectra from the partition function



1d quantum Ising universality class

all spectra match perfectly between the different calculations, since the different models can be **mapped into each other exactly**.

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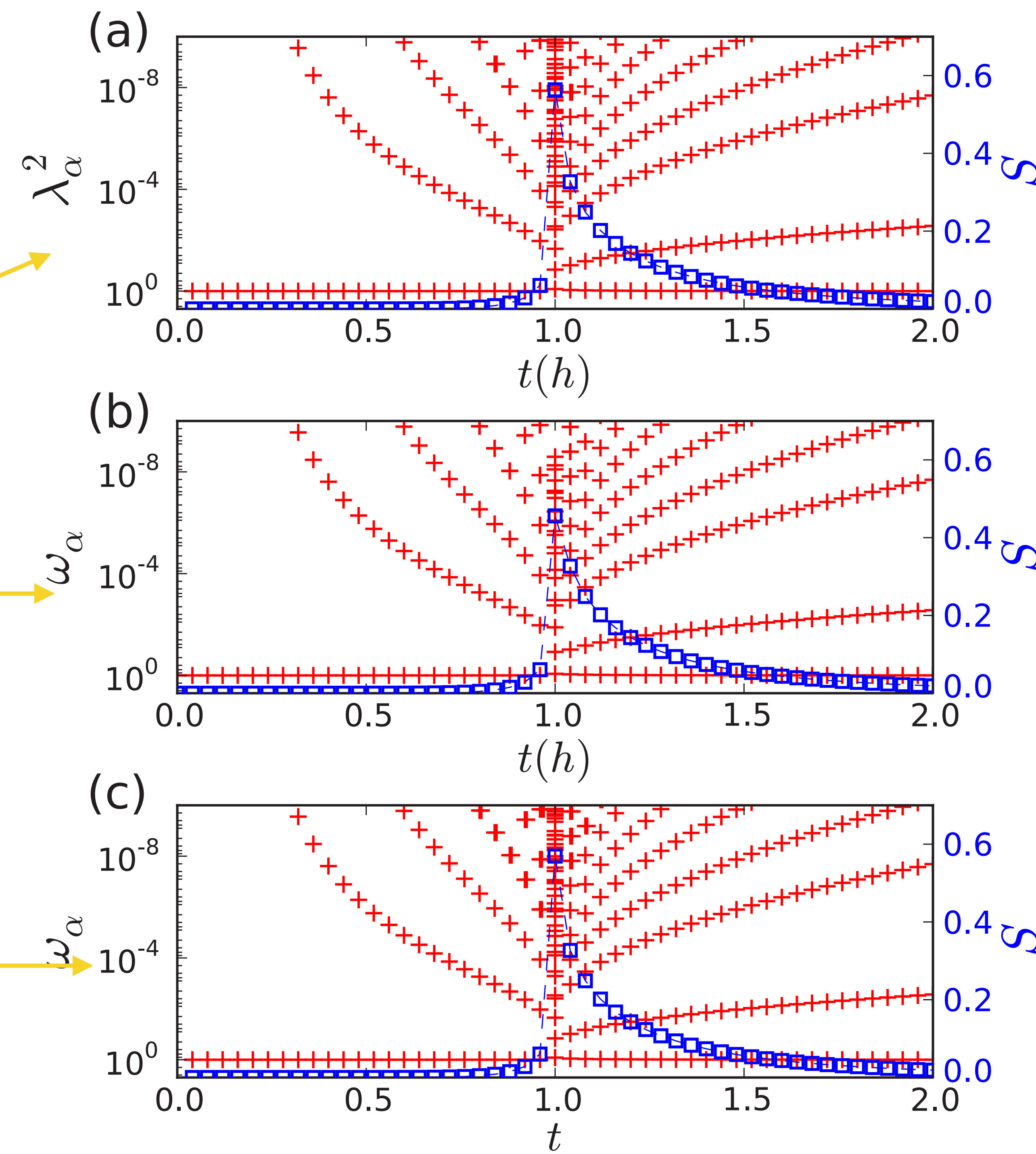

The quantum-classical correspondence

- The corner energies for a variety of quantum and classical systems
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  (a) *The partition-function method*
  (b) Peschel's method
  (c) Suzuki's method

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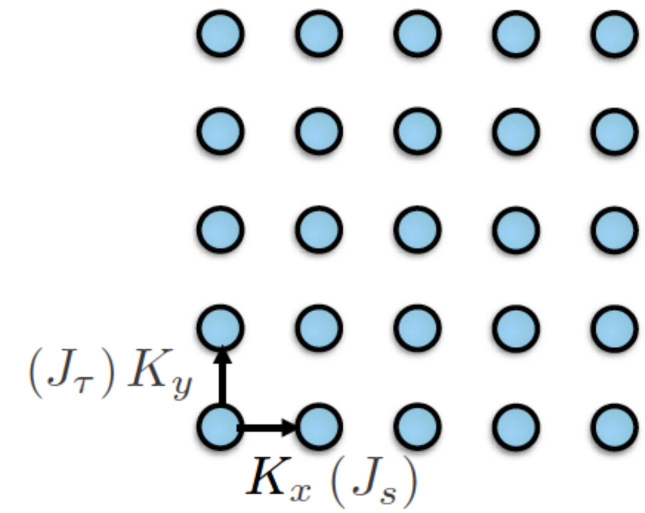
- The main idea is that, for a **d**-dimensional quantum Hamiltonian **Hq** at inverse temperature β, the canonical quantum partition function $Z_q = \mathrm{tr}(e^{-\beta H_q})$ can be evaluated by writing it as a path integral in imaginary time

$$Z_q = \mathrm{tr}(e^{-\beta H_q}) = \sum \langle m | e^{-\beta H_q} | m \rangle,$$

with $|m\rangle$ a given basis of the Hilbert space.

The partition-function method



- Transverse field Ising model in d dimension

$$H_q = -J_z \sum_{\langle i,j \rangle} \sigma_z^{[i]} \sigma_z^{[j]} - J_x \sum_i \sigma_x^{[i]} = H_z + H_x,$$

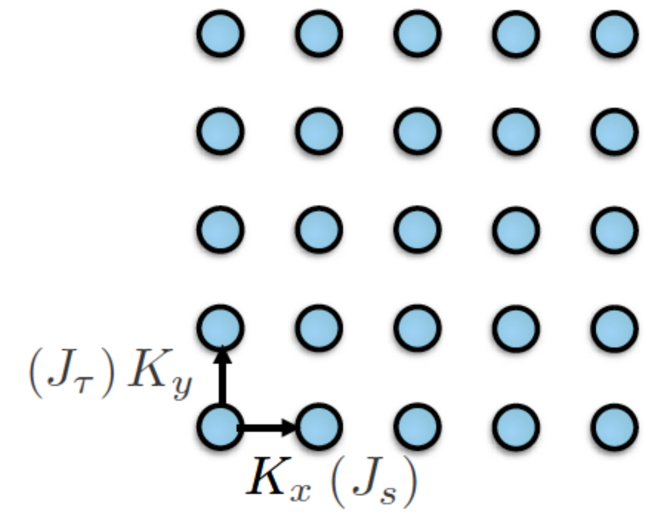
- The canonical quantum partition function of this model

$$Z_q = \mathrm{tr}(e^{-\beta H_q}) = \sum_{\eta_z} \langle \{\eta_z\} | e^{-\beta H_q} | \{\eta_z\} \rangle,$$

- Splitting the imaginary time β into infinitesimal time step $\delta\tau$

$$\begin{aligned} &\langle \{\eta_z(\tau + \delta\tau)\} | e^{-\delta\tau H_q} | \{\eta_z(\tau)\} \rangle \\ &\quad \approx \langle \{\eta^z(\tau + \delta\tau)\} | e^{-\delta\tau H_x} e^{-\delta\tau H_z} | \{\eta^z(\tau)\} \rangle \\ &\quad = e^{-\delta\tau H_z(\{\eta_z(\tau)\})} \langle \{\eta^z(\tau + \delta\tau)\} | e^{-\delta\tau H_x} | \{\eta^z(\tau)\} \rangle, \end{aligned}$$

The partition-function method



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$$Z_q \approx \sum_{\{\eta\}} C' e^{J_s \sum_{\alpha,\langle i,j\rangle} \eta_z^{[i]}(\tau_\alpha) \eta_z^{[j]}(\tau_\alpha)} e^{J_\tau \sum_{\alpha,i} \eta_z^{[i]}(\tau_{\alpha+1}) \eta_z^{[i]}(\tau_\alpha)},$$

where the "coupling constants" along the imaginary-time (τ) and space (s) directions are given by

$$J_\tau = \tanh^{-1}(e^{-2\delta\tau J_x}) \quad J_s = J_z \delta\tau.$$

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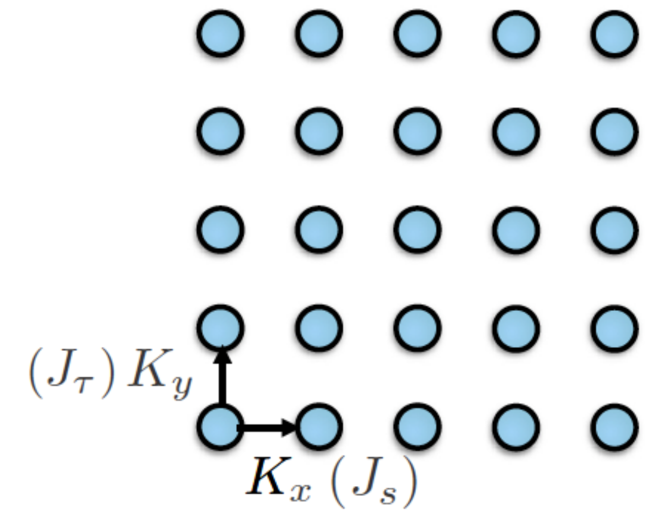
The partition-function method



- Transverse field Ising model in d dimension

$$H_q = -J_z \sum_{\langle i,j\rangle} \sigma_z^{[i]}\sigma_z^{[j]} - J_x \sum_i \sigma_x^{[i]} = H_z + H_x,$$

- The canonical quantum

$$Z_q = \mathrm{tr}(e^{-\beta H_q}) = \sum_{\eta_z} \langle\{\eta_z\}|e$$

- Splitting the imaginary t

$$\langle\{\eta_z(\tau+\delta\tau)\}|e^{-\delta\tau H_q}|\{\eta_z(\tau)\}\rangle$$
$$\approx \langle\{\eta^z(\tau+\delta\tau)\}|e^{-\delta\tau H_x}e^{-\delta\tau H_z}|\{\eta^z(\tau)\}\rangle$$
$$= e^{-\delta\tau H_z(\{\eta_z(\tau)\})}\langle\{\eta^z(\tau+\delta\tau)\}|e^{-\delta\tau H_x}|\{\eta^z(\tau)\}\rangle,$$

The exact correspondence arrives if we take the number of sites L in the imaginary time direction to be infinity, giving $\delta = \beta/L \to 0$, and then the corresponding classical model has the couplings $J_s \to 0$ and $J_\tau \to \infty$.

$$Z_q \approx \sum_{\{\eta\}} C' e^{J_s \sum_{\alpha,\langle i,j\rangle} \eta_z^{[i]}(\tau_\alpha)\eta_z^{[j]}(\tau_\alpha)} e^{J_\tau \sum_{\alpha,i} \eta_z^{[i]}(\tau_{\alpha+1})\eta_z^{[i]}(\tau_\alpha)},$$

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The quantum-classical correspondence

- 1d quantum Ising model:

$$H_q = -\sum_i \sigma_x^{[i]}\sigma_x^{[i+1]} - h\sum_i \sigma_z^{[i]},$$

(a) Entanglement spectra and the entanglement entropy obtained from iTEBD

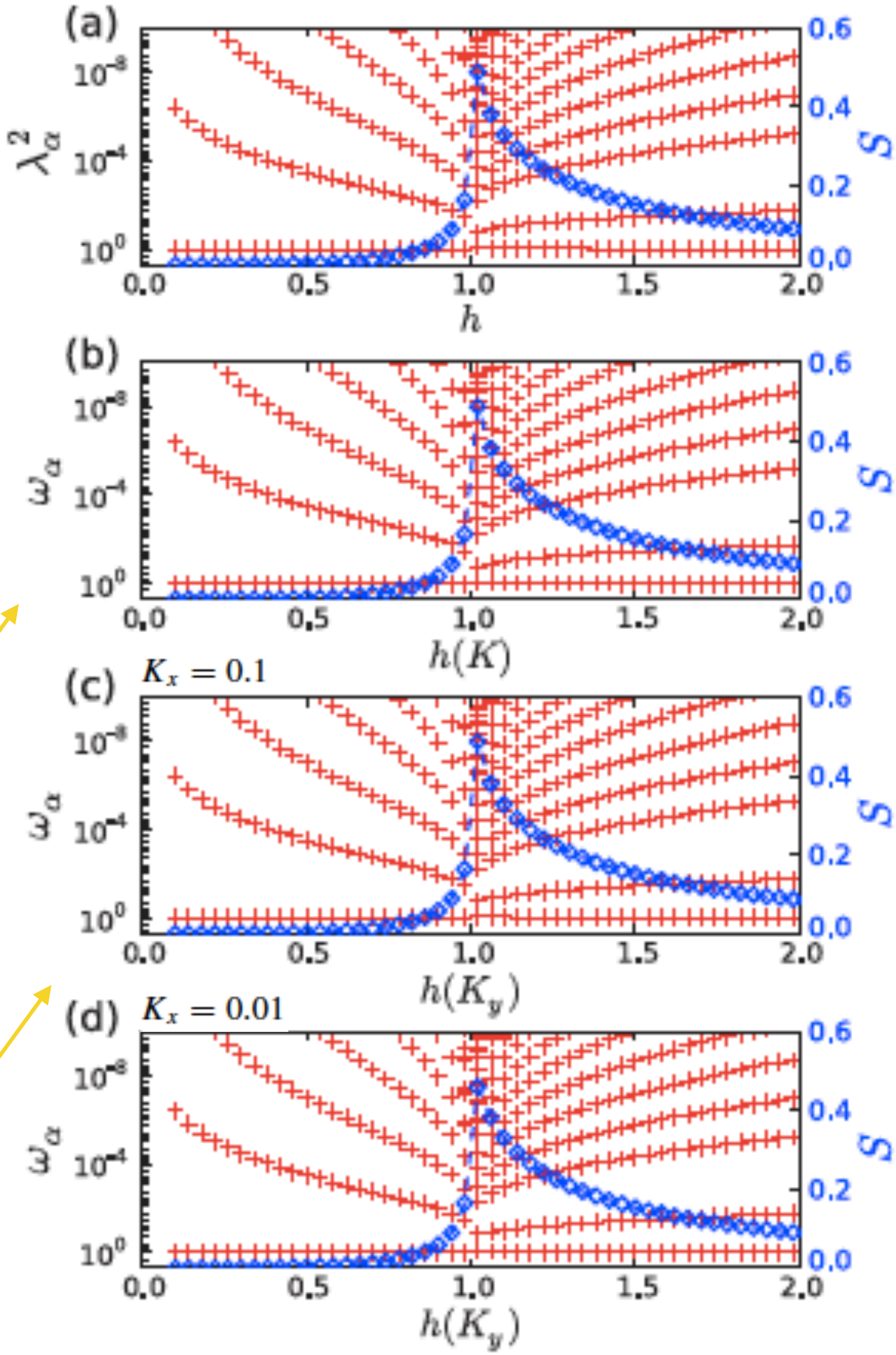
- 2d classical isotropic Ising model

$1/h = \sinh^2 K$

(b) To compute the corner spectra from the partition function

- 2d classical anisotropic Ising model

(c) (d) To compute the corner spectra from the partition function



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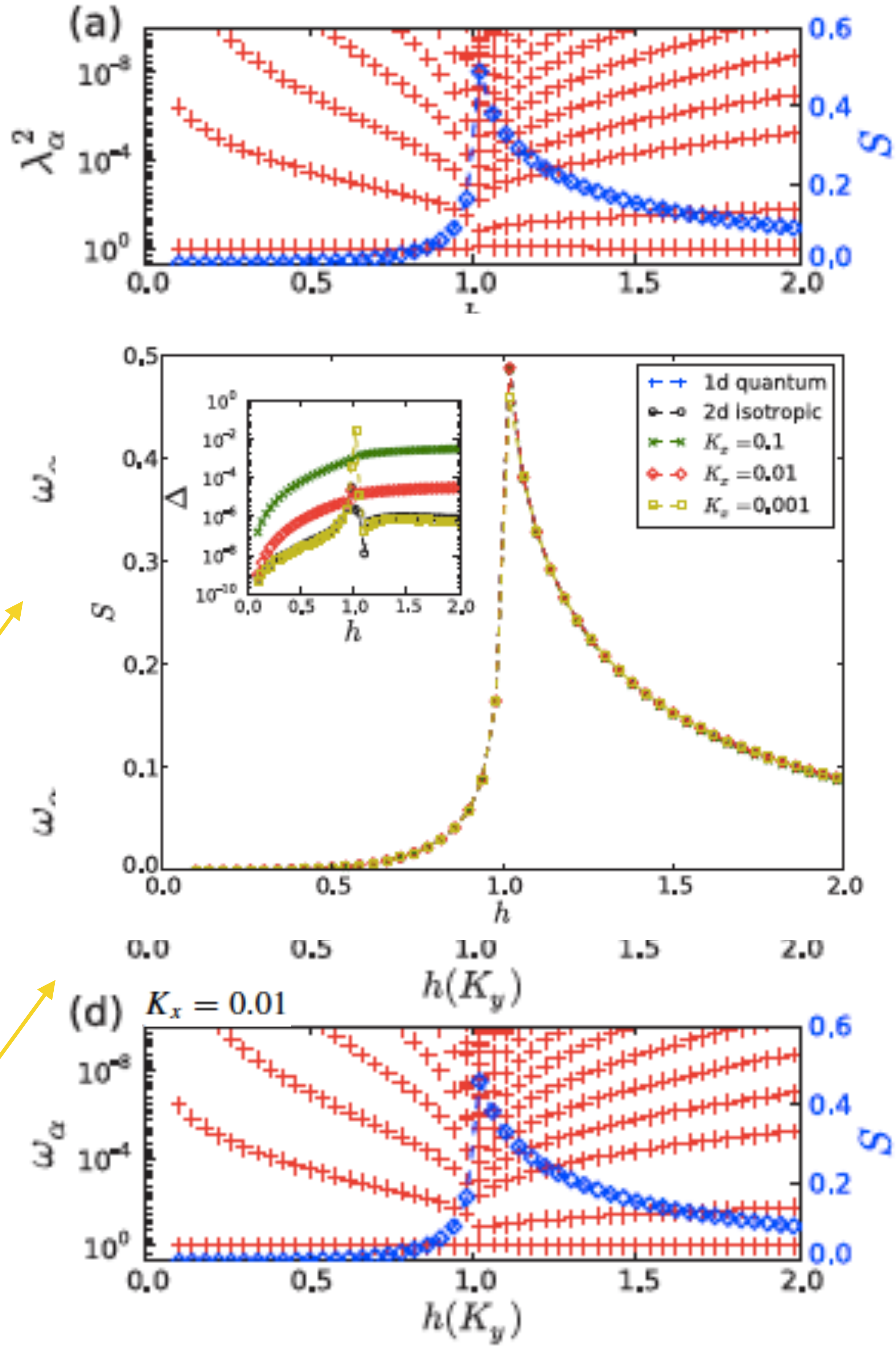


2d quantum Ising model

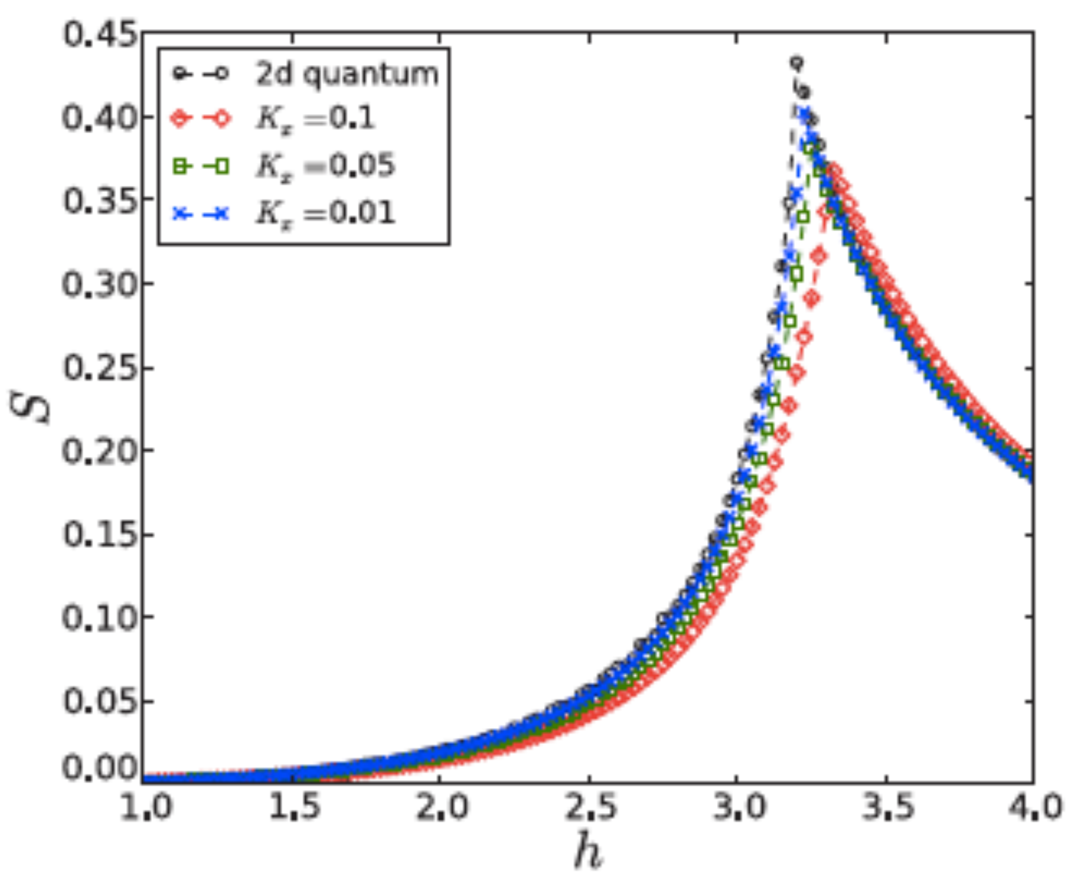
- 2d quantum Ising model:

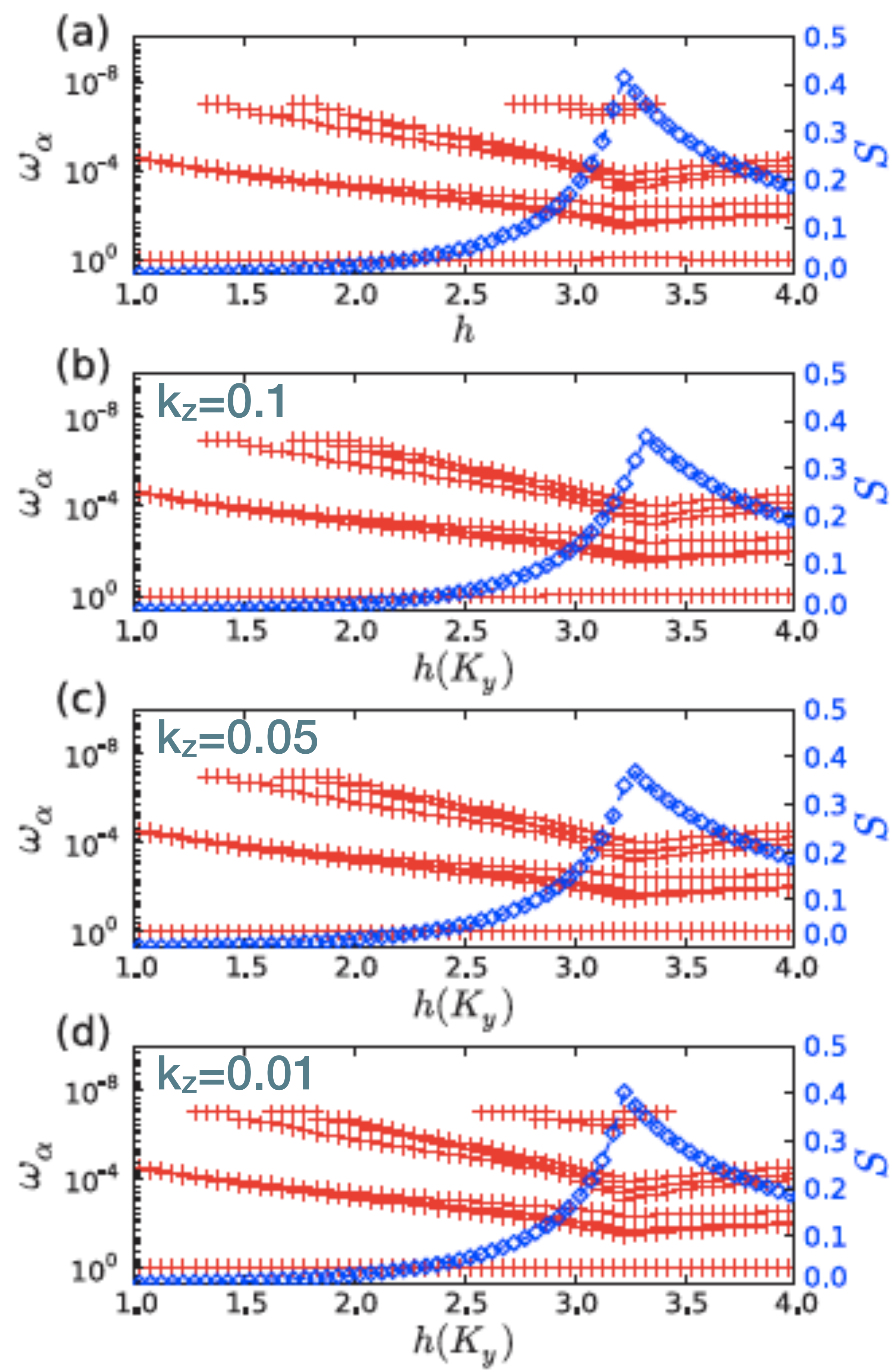
$$H_q = -\sum_i \sigma_x^{[i]}\sigma_x^{[i+1]} - h\sum_i \sigma_z^{[i]},$$

- 3d classical anisotropic Ising model

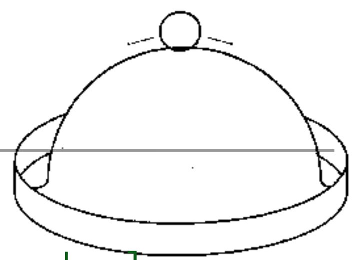
$$K_x = J_s = J_z\delta\tau, \quad K_y = J_s = J_z\delta\tau,$$

$$K_z = J_\tau = \tanh^{-1}(e^{-2\delta\tau J_z}).$$





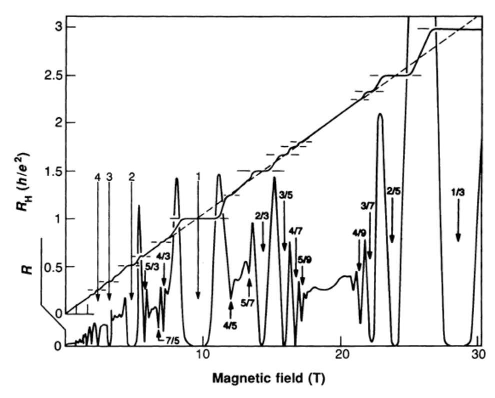
Phase of matter



[Landau]

- Conventional phases of matter: understood through spontaneous symmetry-breaking
  => Local order parameters: distinguish different phases

- New phases of matter: e.g. Fractional quantum Hall effect
  No local order parameters
  No symmetry breaking



[Tsui, Stormer, & Gossard '82]

intrinsic Topological Order	**Symmetry protected topological order**
2D Z_2 Toric code	1D Haldane phase
Ground state degeneracy	NO
Fractional statistics of quasiparticles	NO
Topological entanglement entropy	NO
Long range entanglement	*Short range entanglement*

Symmetry-protected topological order

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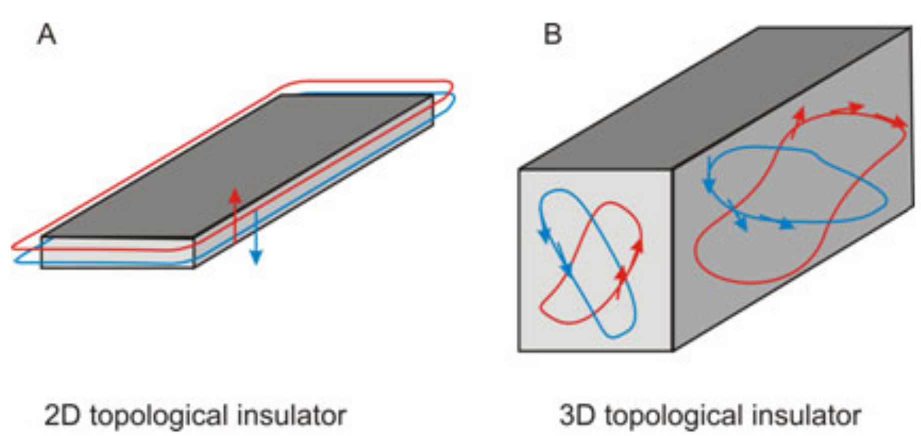
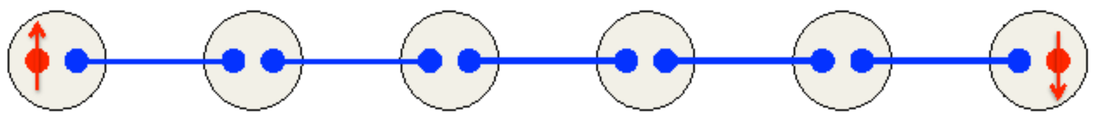
- *Unique ground state* on periodic boundary, Usually *gapless edge excitations*
- Example: Topological insulator, Haldane phase in spin-1 chain



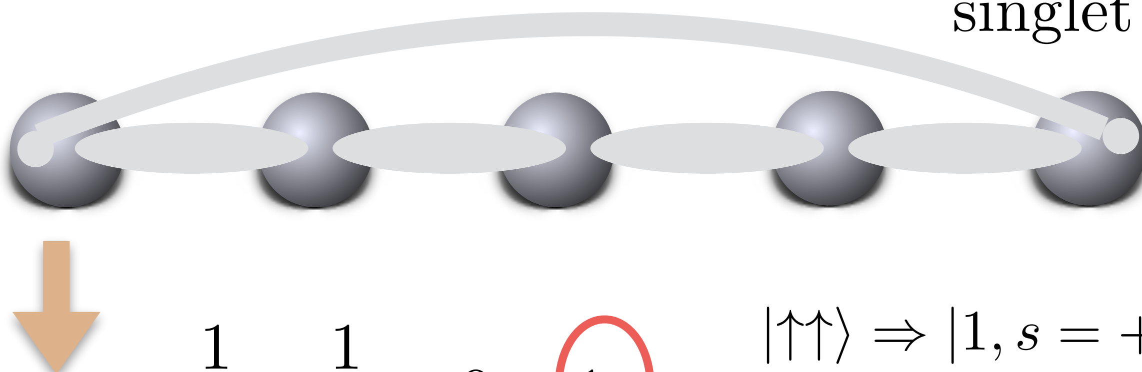


Prominent example of SPT state:
1D Affleck-Kennedy-Lieb-Tasaki state

- Each site: spin-1

$$\text{singlet} = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

[*AKLT* '87,88]



Project into symmetric subspace of two spin-1/2 (qubits)

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

$$|\uparrow\uparrow\rangle \Rightarrow |1, s = +1\rangle$$

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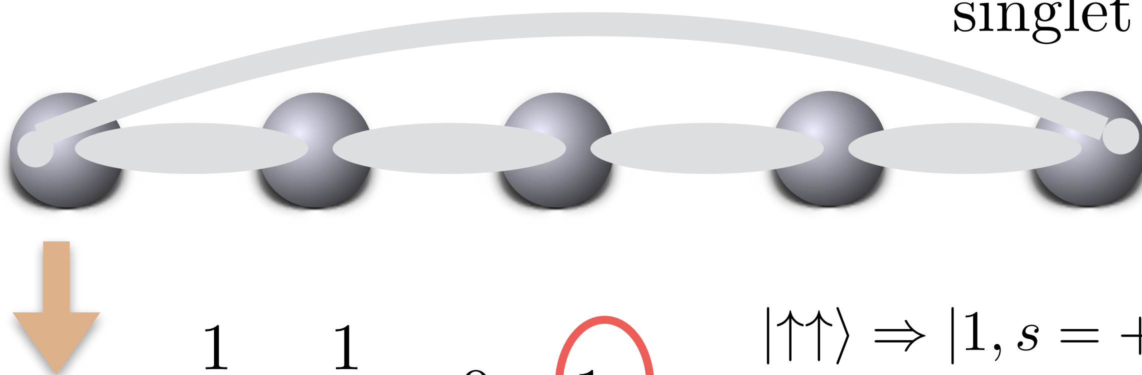

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- Unique ground state on periodic chain; 4 gapless edge states if open

$$H = \sum_i [\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{2}{3}] = 2\sum_i \hat{P}^{(S=2)}_{i,i+1}$$

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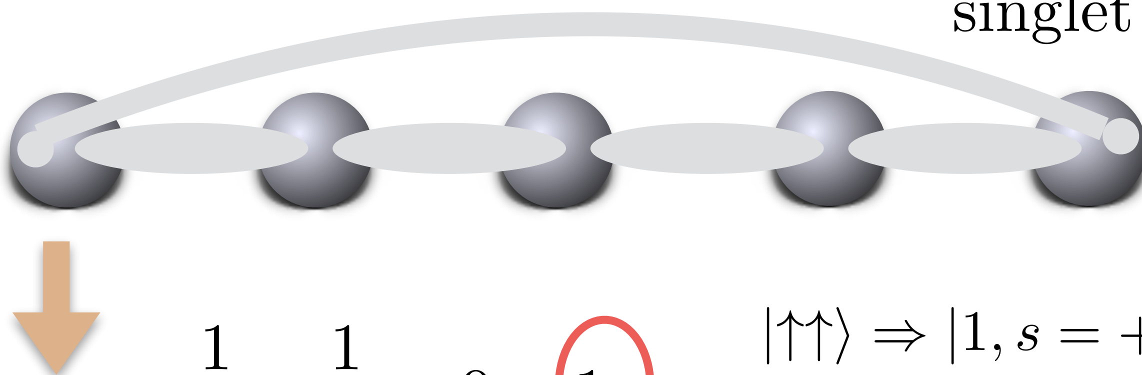

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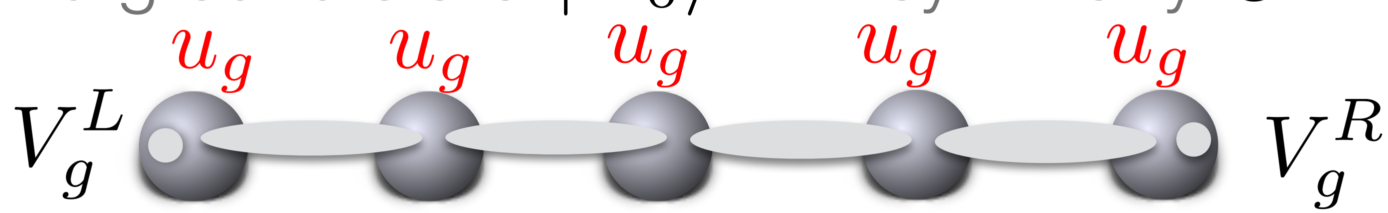
- SO(3) fractionalizes to SU(2) (i.e. spin-1/2 representation) at open ends ➔ projective representation as signature of 1D SPT

Characterization of 1D SPT phases

- Hamiltonian and ground state $|\Psi_0\rangle$ with symmetry G

$U_g|\Psi_0\rangle$



- bulk: Linear on-site representation $U_g U_h = U_{gh}$ (e.g. spin-1)
- boundary: Projective representation V_g (e.g. spin 1/2)

- A projective representation respects group multiplication

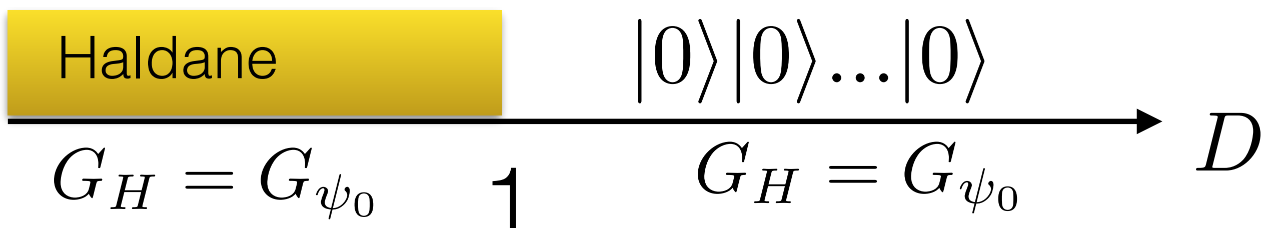
$$V_g V_h = \omega(g,h) V_{gh}$$

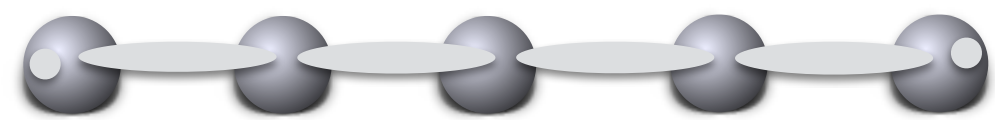
$\omega(g,h)$ → U(1) phase [i.e. 2-cocycle]

- 1D nontrivial SPT phases are characterized by projective representation of symmetry action at one end
- Classified by the second cohomology group $H^2[G, U(1)]$

Example: spin-1 chain

$$H = \sum_i [\vec{S}_i \cdot \vec{S}_{i+1} + D(S_i^z)^2]$$





$$U(g) = [u(g)]^{\otimes N}, \ g \in G$$
$$U(g_1)U(g_2) = U(g_1 g_2)$$

- Rx, Rz rotation symmetry: Rx Rz =Rz Rx
- Haldane phase

  Rx, Rz rotation symmetry represented by s=1/2 Pauli matrix

  $$\sigma_x \sigma_z = -\sigma_z \sigma_x \Rightarrow \omega = -1$$

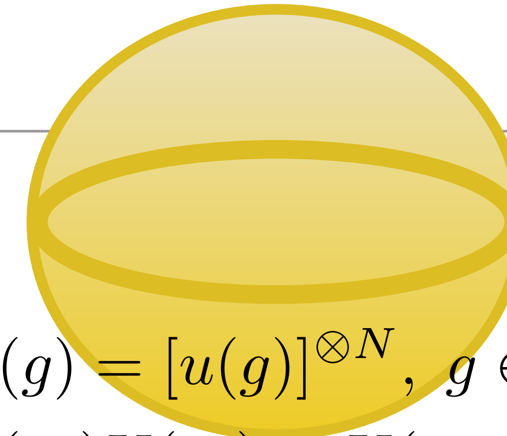
- Large D phase

  Rx, Rz rotation symmetry represented by $\mathbb{I} \Rightarrow \omega = 1$

- The second cohomology group $H^2[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2$

2D SPT phases: characterization

- Characterized by obstruction of symmetry action on boundary with open ends



$U(g) = [u(g)]^{\otimes N}, \; g \in G$

$U(g_1)U(g_2) = U(g_1 g_2)$

- on closed 2d manifold: $U(g)|\psi\rangle = |\psi\rangle$

- open 2d manifold →symmetry action on boundary C

C

$U_c(g)|\psi_c\rangle = |\psi_c\rangle$

- consider region M of C → symmetry action

$$U_M(g_1)U_M(g_2) = \Omega(g_1, g_2)U_M(g_1, g_2)$$

- Associativity → 3-cocycle [a U(1) phase]

$$\Omega_a(g_1, g_2)\Omega_a(g_1 g_2, g_3) = \phi(g_1, g_2, g_3)\Omega_a(g_2, g_3)\Omega_a(g_1, g_2 g_3)$$

3-cocycle (a U(1) phase)

2d SPT phases: CZX model

- CZX model: nontrivial SPT order protected only by on-site Z2 symmetry.

- On site Z2 symmetry:

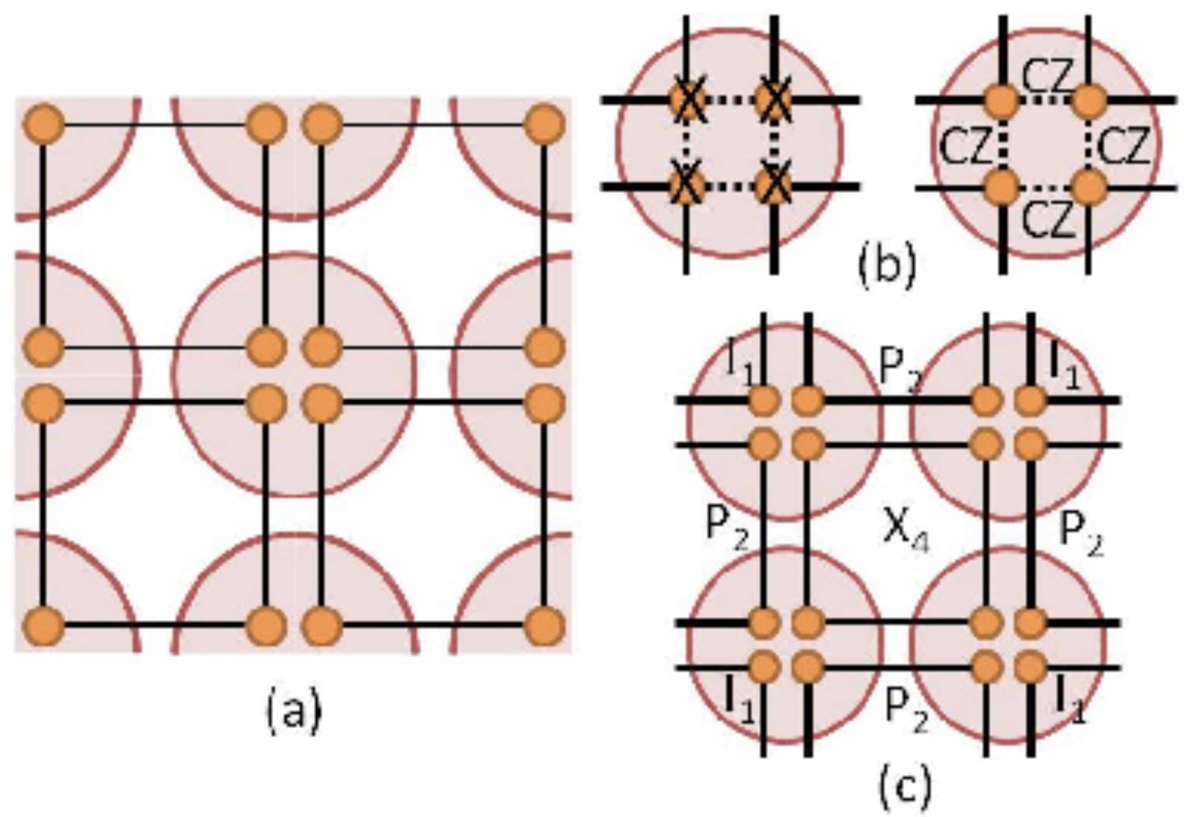
$$U_{CZX} = U_X U_{CZ}$$
$$U_X = X_1 \otimes X_2 \otimes X_3 \otimes X_4$$
$$U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}$$

- The Hamiltonian:

$$H_{p_i} = -X_4 \otimes P_2^u \otimes P_2^d \otimes P_2^l \otimes P_2^r$$
$$X_4 = |0000\rangle\langle 1111| + |1111\rangle\langle 0000|$$
$$P_2 = |00\rangle\langle 00| + |11\rangle\langle 11|$$

- The ground state: every four spins around a plaquette are entangled in the state

$$|\psi_{pl}\rangle = |0000\rangle + |1111\rangle$$



[Chen & Wen 2012]

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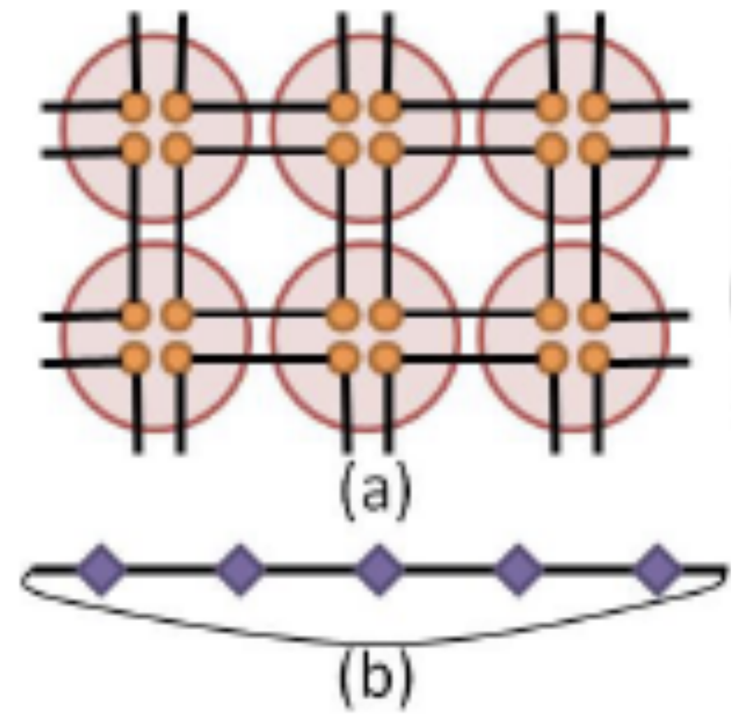
- non-trivial edge state

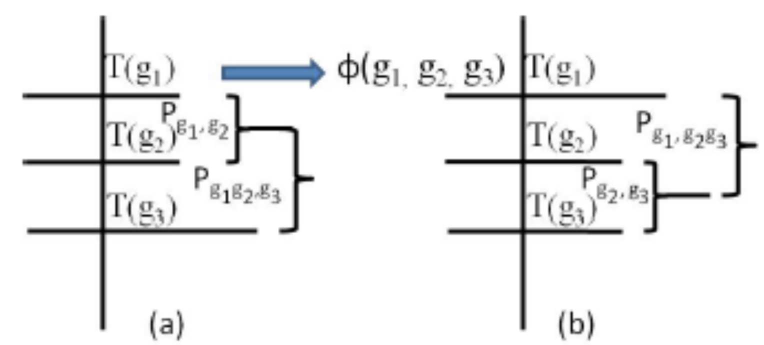


- The effective symmetry action on the boundary of CZX model can be expressed as MPO

$$T^{0,1}(CZX) = |0\rangle\langle +|, \quad T^{1,0}(CZX) = |1\rangle\langle -|, \quad \text{other terms are zero}$$

$$T^{0,0}(I) = |0\rangle\langle 0|, \quad T^{1,1}(I) = |0\rangle\langle 0|, \quad \text{other terms are zero}$$

$$T_{g_1} T_{g_2} = P(g_1, g_2) T_{g_1 g_2}$$



$$P_{g_1,g_2} P_{g_1 g_2, g_3} = \phi(g_1, g_2, g_3) P_{g_2,g_3} P_{g_1, g_2 g_3}$$

- 3-cocylce for the group generated by UCZX

$$\phi(I,I,I) = 1 \quad \phi(I,I,CZX) = 1$$
$$\phi(I,CZX,I) = 1 \quad \phi(CZX,I,I) = 1$$
$$\phi(I,CZX,CZX) = 1 \quad \phi(CZX,CZX,I) = 1$$
$$\phi(CZX,I,CZX) = 1 \quad \phi(CZX,CZX,CZX) = -1$$

nontrivial 3-cocycle for the Z_2 group

Classification of (symmetry protected) topological order phase

- For bosonic system: Topological order
  → **Tensor category**

  Symmetry protected topological order
  → **Group cohomology**

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_n \times \text{trn}$	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$U(1) \times \text{trn}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \rtimes Z_2^T \times \text{trn}$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1) \times Z_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_1$
$Z_n \rtimes Z_2^T$	$\mathbb{Z}_n$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \times Z_2^T$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \rtimes Z_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$
$Z_m \times Z_n$	$\mathbb{Z}_m \times \mathbb{Z}_n$	$\mathbb{Z}_{(m,n)}$	$\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}^2$
$D_2 \times Z_2^T = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$Z_m \times Z_n \times Z_2^T$	$\mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)} \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(2,m,n)}^2 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m,n)}^4 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$
$SU(2)$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}^3 \times \mathbb{Z}_2^3$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$SO(3) \times Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^{12}$

[Chen, Gu,Liu & Wen 2013]

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Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
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$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \rtimes Z_2^T \times \text{trn}$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
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$U(1) \times Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1) \times Z_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_1$
$Z_n \rtimes Z_2^T$	$\mathbb{Z}_n$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \times Z_2^T$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \rtimes Z_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$
$Z_m \times Z_n$	$\mathbb{Z}_m \times \mathbb{Z}_n$	$\mathbb{Z}_{(m,n)}$	$\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}^2$
$D_2 \times Z_2^T = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$Z_m \times Z_n \times Z_2^T$	$\mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)} \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(2,m,n)}^2 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m,n)}^4 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$
$SU(2)$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}^3 \times \mathbb{Z}_2^3$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$SO(3) \times Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^{12}$

[Chen, Gu,Liu & Wen 2013]

- Question: Numerically, how to detect different topological order phases and phase transition?

Classification of (symmetry protected) topological order phase

- For bosonic system:
  Topological order
  → **Tensor category**

  Symmetry protected
  topological order
  → **Group cohomology**

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_n \times \text{trn}$	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$U(1) \times \text{trn}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \rtimes Z_2^T \times \text{trn}$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times \text{trn}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1) \times Z_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_1$
$Z_n \rtimes Z_2^T$	$\mathbb{Z}_n$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \times Z_2^T$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^2$
$Z_n \rtimes Z_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$
$Z_m \times Z_n$	$\mathbb{Z}_m \times \mathbb{Z}_n$	$\mathbb{Z}_{(m,n)}$	$\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}^2$
$D_2 \times Z_2^T = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$Z_m \times Z_n \times Z_2^T$	$\mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m)} \times \mathbb{Z}_{(2,n)} \times \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(2,m,n)}^2 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{(2,m,n)}^4 \times \mathbb{Z}_{(2,m)}^2 \times \mathbb{Z}_{(2,n)}^2$
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$\mathbb{Z}_2$ $\mathbb{Z}_2$

[Chen, Gu,Liu & Wen 2013]

- Question: Numerically, how to detect different topological order phases and phase transition?

Transition between SPT

Transition between SPT

- What is transition between two SPT^k phases?

Transition between SPT

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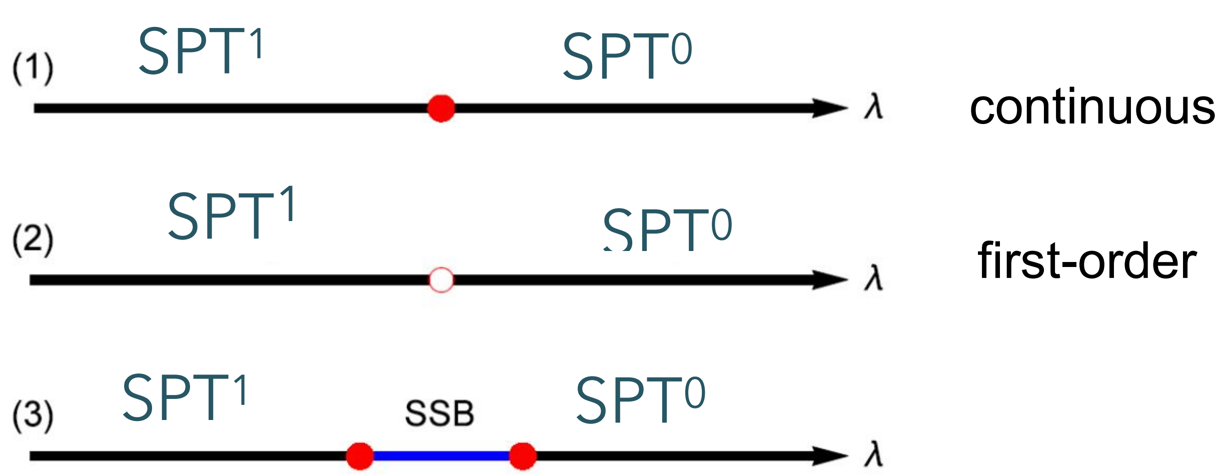
- Three scenarios between two SPT phases



[Tsui,Jiang,Lu& Lee 2015]

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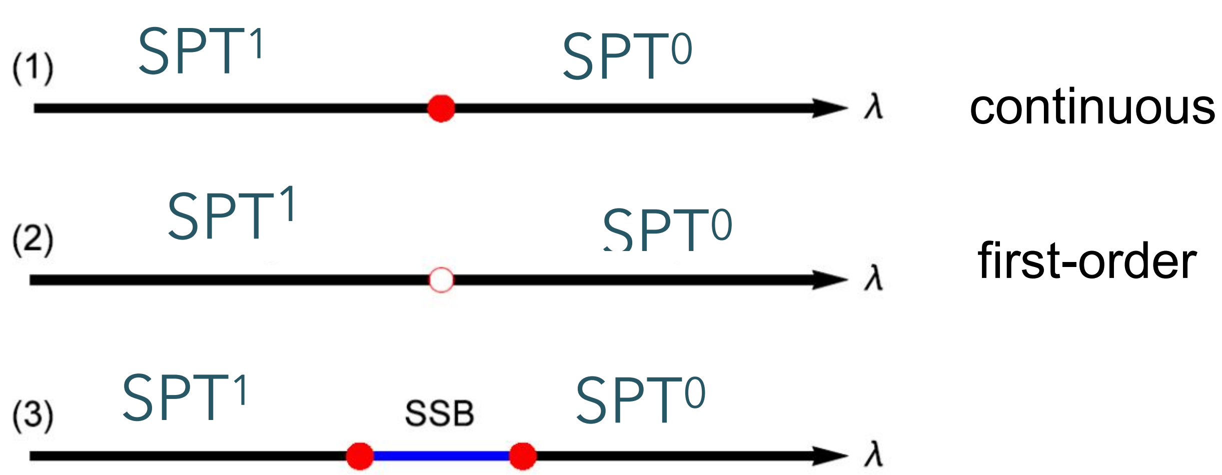

[Tsui,Jiang,Lu& Lee 2015]

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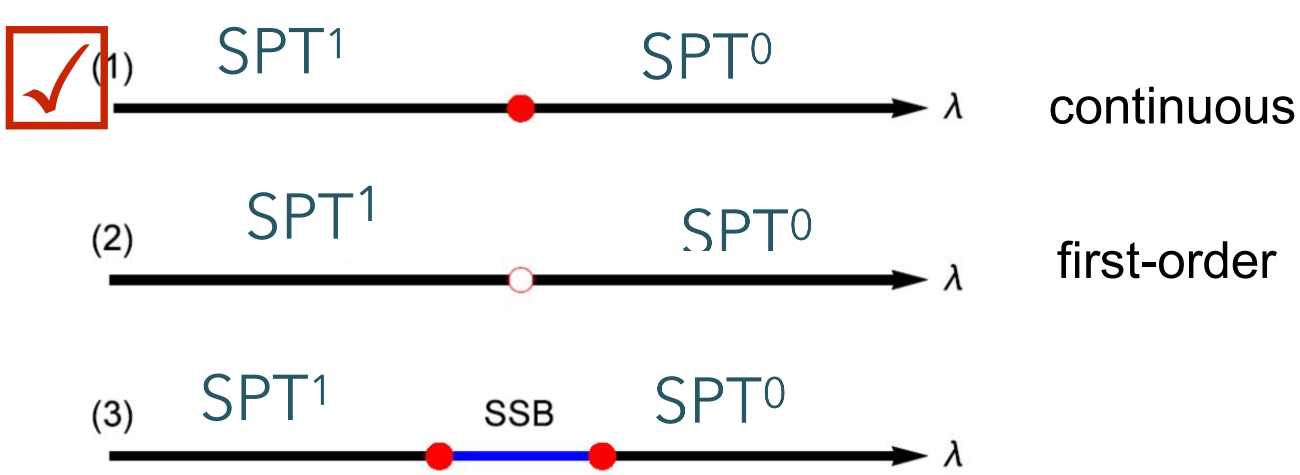


[Tsui,Jiang,Lu& Lee 2015]

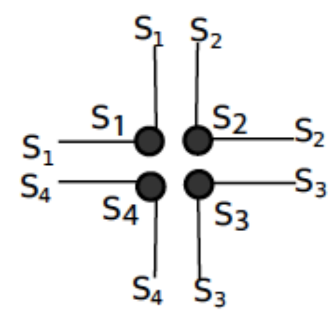
- To study phase transition from the corner structure of the norm of quantum state

Transition between SPT

- The wave function

$$|\Psi\rangle = \sum_{s_i} tTr(A \otimes A... \otimes A)|s_1, s_2, ...\rangle.$$

where $A[s_i,s_j,s_k,s_l]$ is a tensor



- The Z_2 SPT^k (t=+1,-1) wave function

$$A[0,0,0,0] = A[1,1,1,1] = A[0,0,1,1] = A[1,1,0,0] = 1,$$
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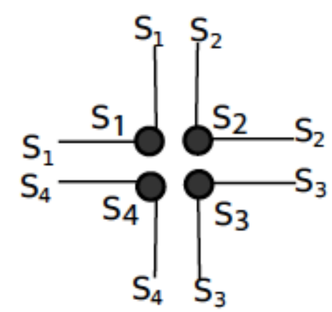

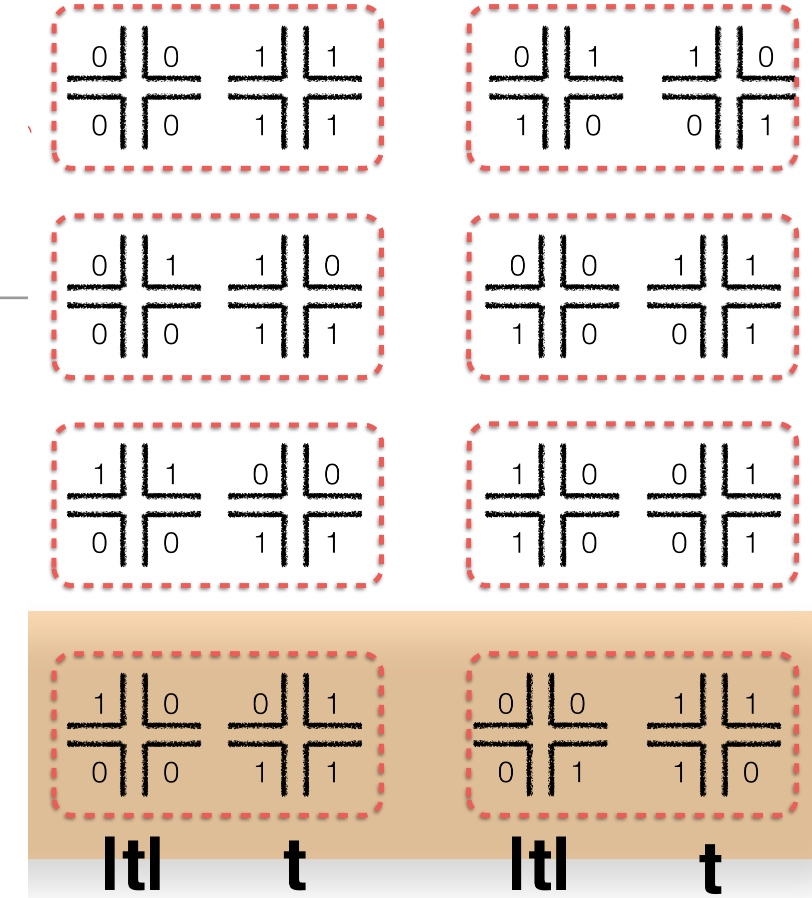

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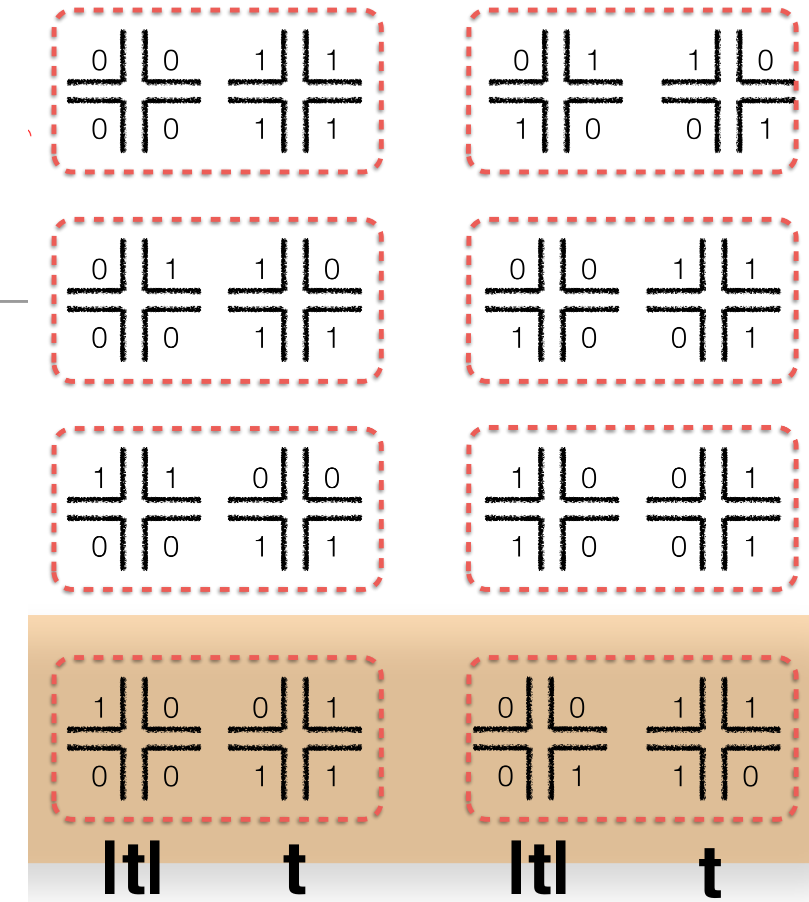

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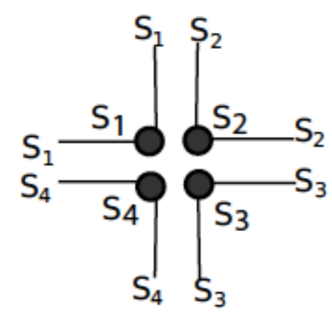

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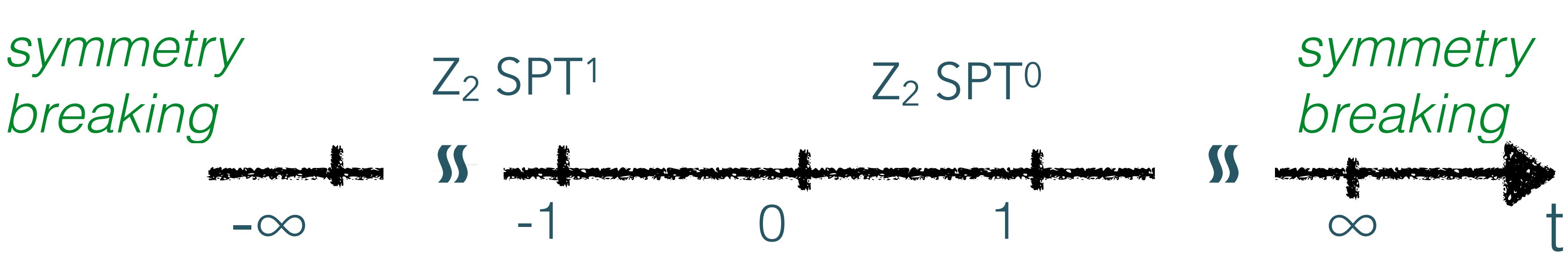

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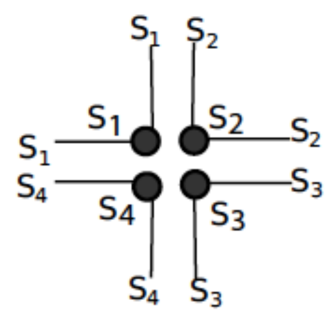

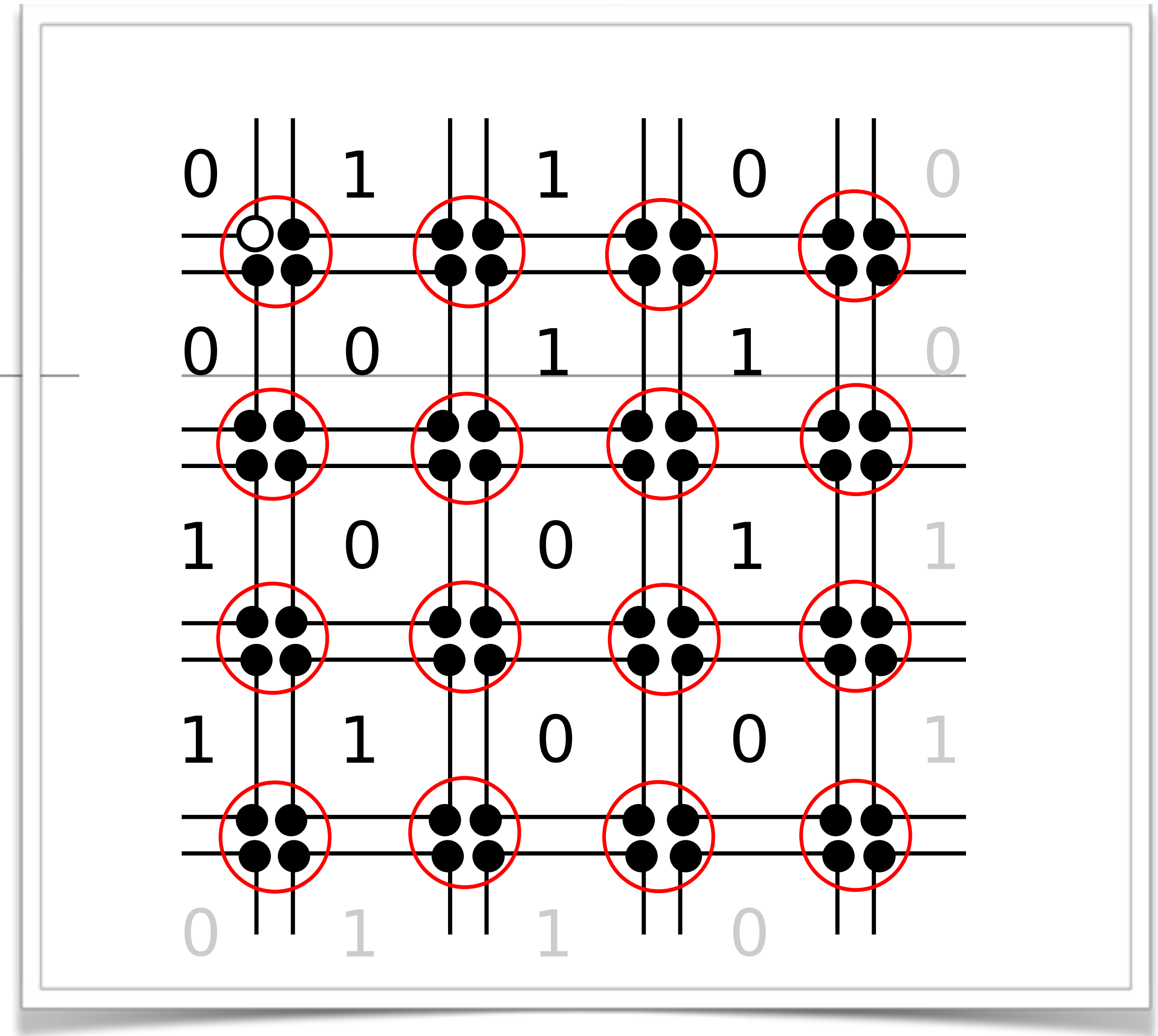

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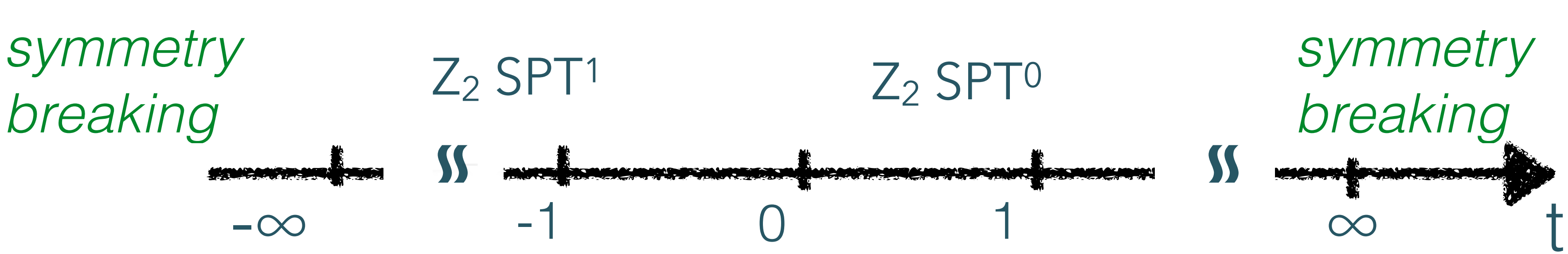

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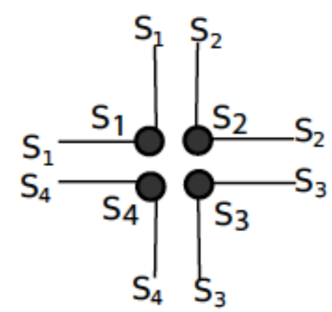

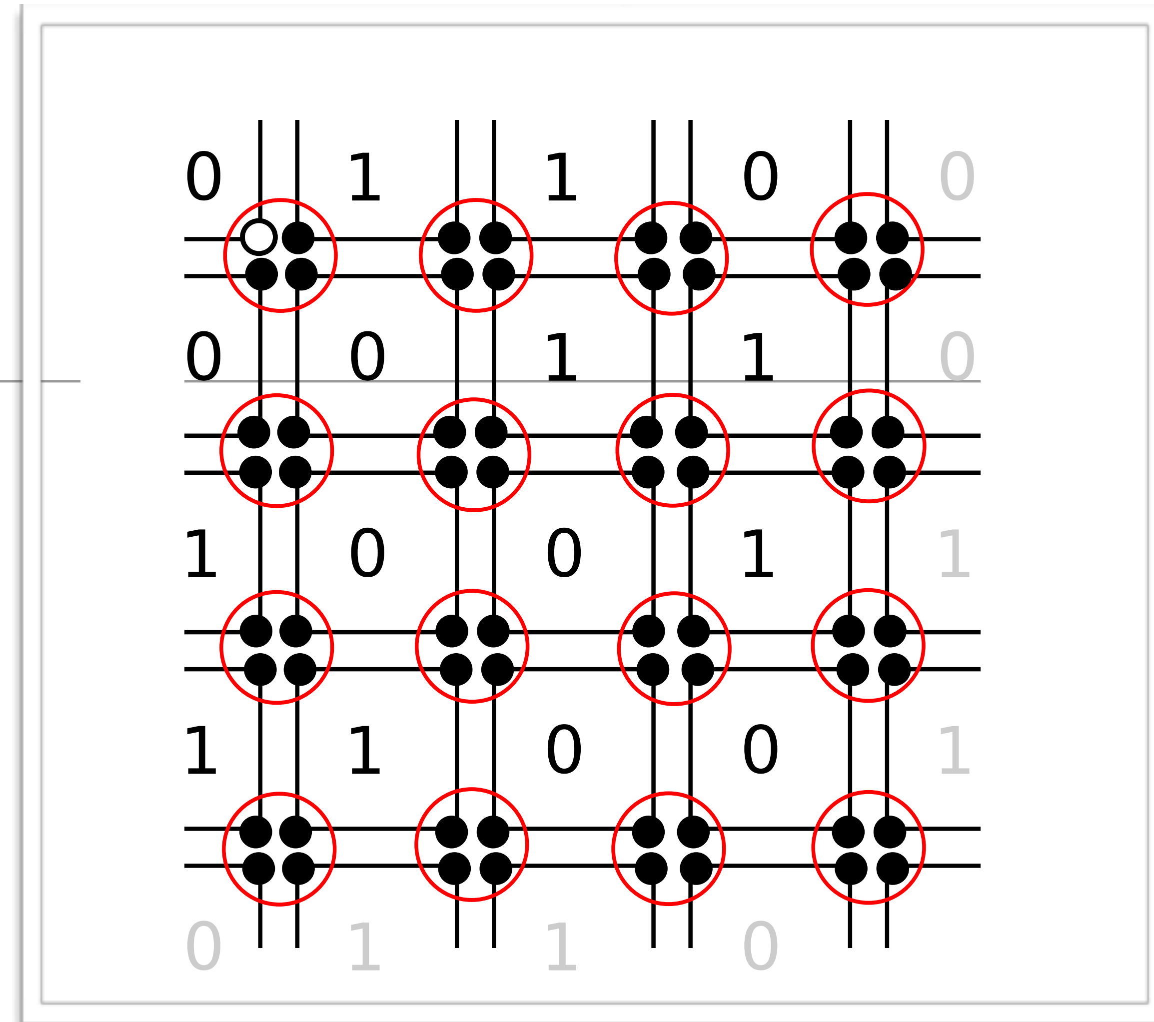

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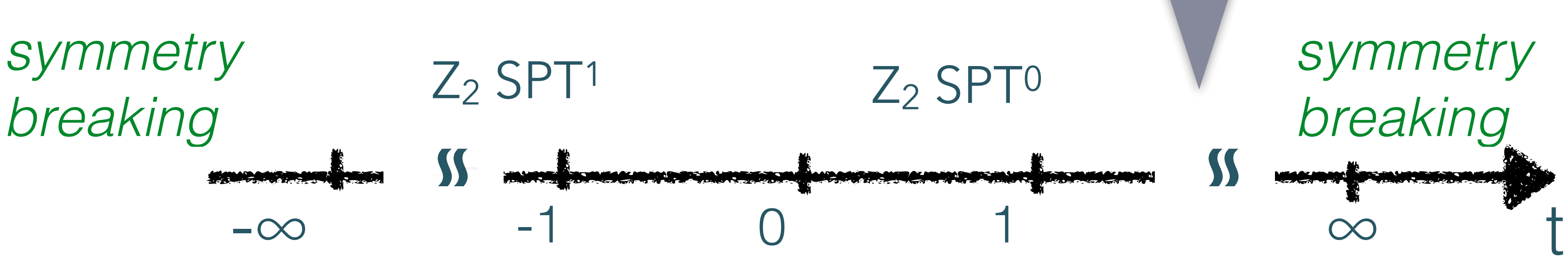
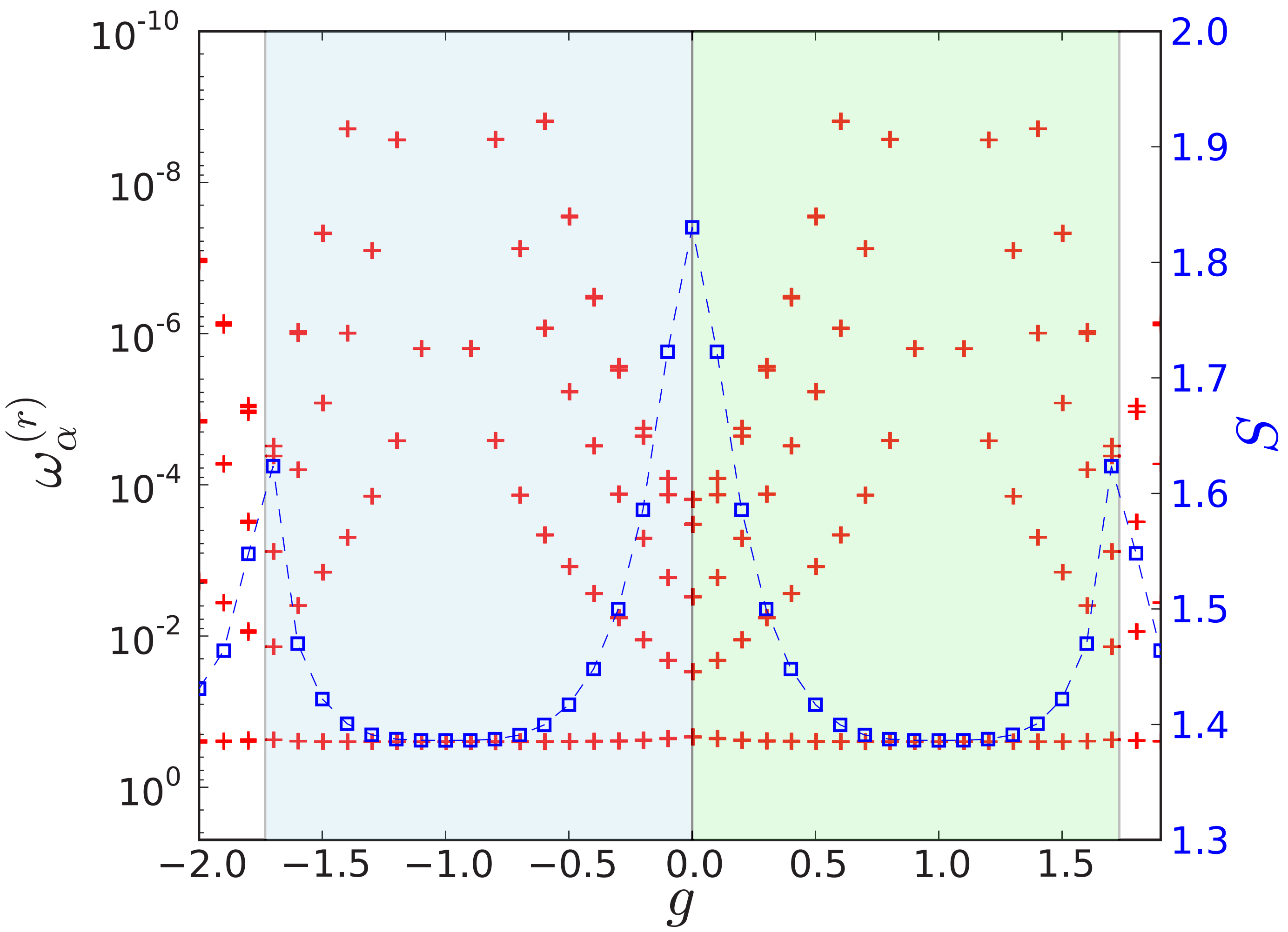
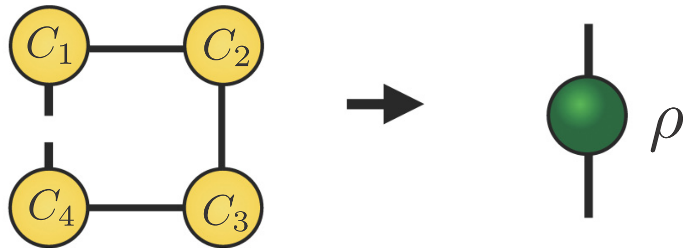
Where is the transition point?



2d corner phase transition

- The reduced corner spectra and entropy of the double-layer tensor defining the norm via the directional CTM approach





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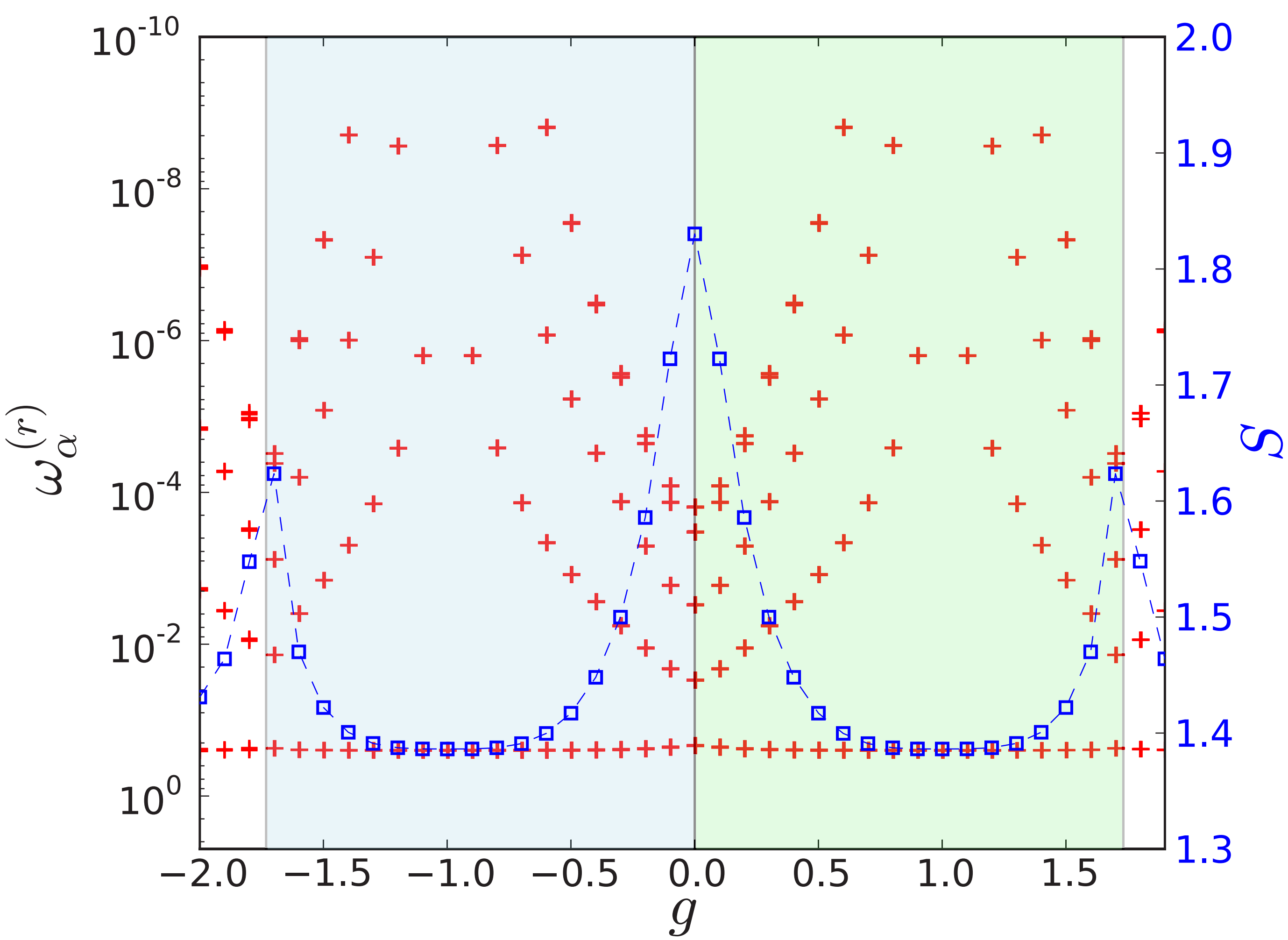
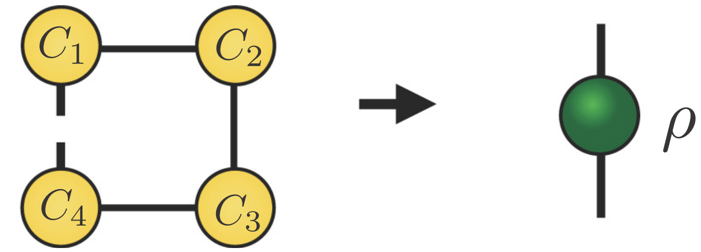
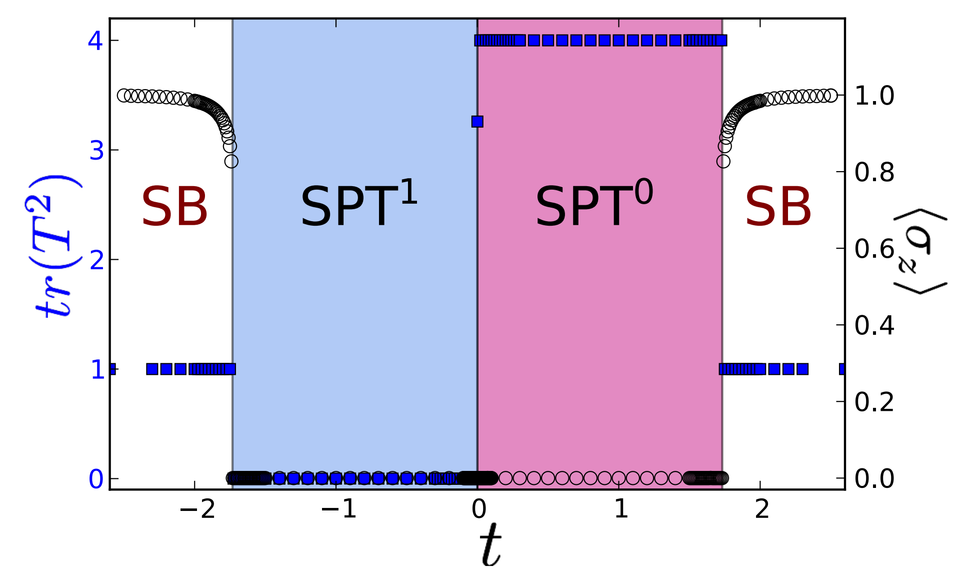




- local order parameter and modular T matrix

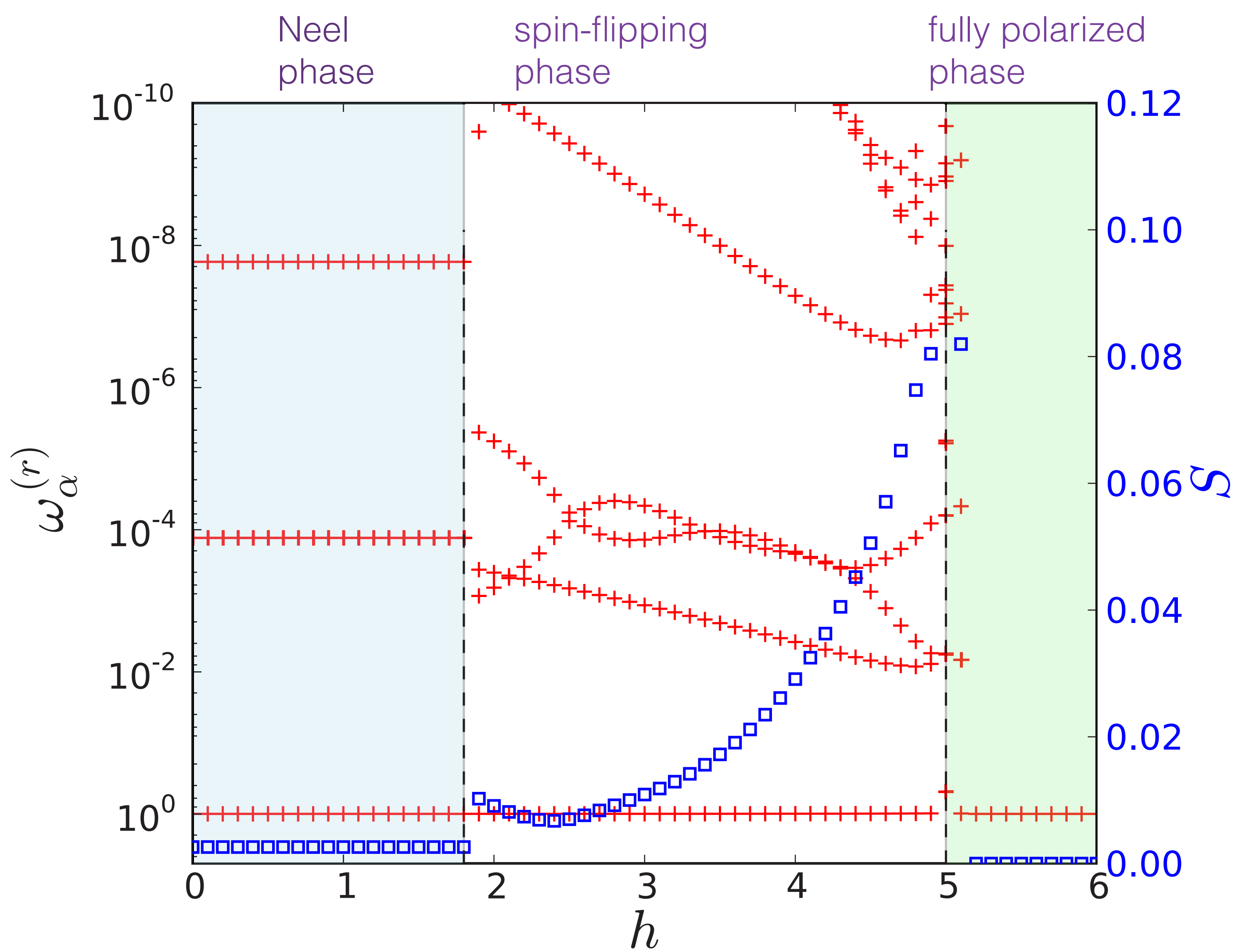


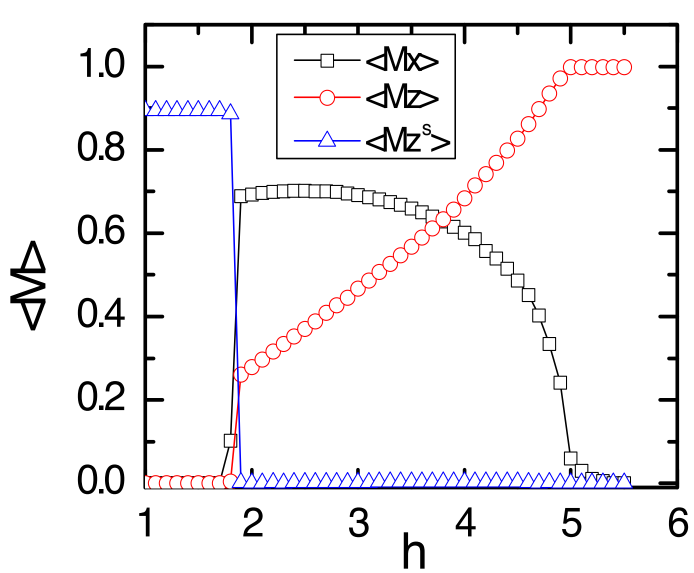
[**C.-Y. Huang** and Tzu-Chieh Wei 2016]

2d corner phase transition

- 2d quantum XXZ model in a uniform z-axis magnetic field
- A *first-order* spin-flop quantum phase transition from **Neel** to **spin-flipping phase**
- Another critical value at hs = 2(1 + Δ), the **fully polarized state** is reached.

$$H_q = -\sum_{\langle i,j \rangle} \left(\sigma_x^{[i]}\sigma_x^{[j]} + \sigma_y^{[i]}\sigma_y^{[j]} - \Delta\sigma_z^{[i]}\sigma_z^{[j]} \right) - h\sum_i \sigma_z^{[i]},$$

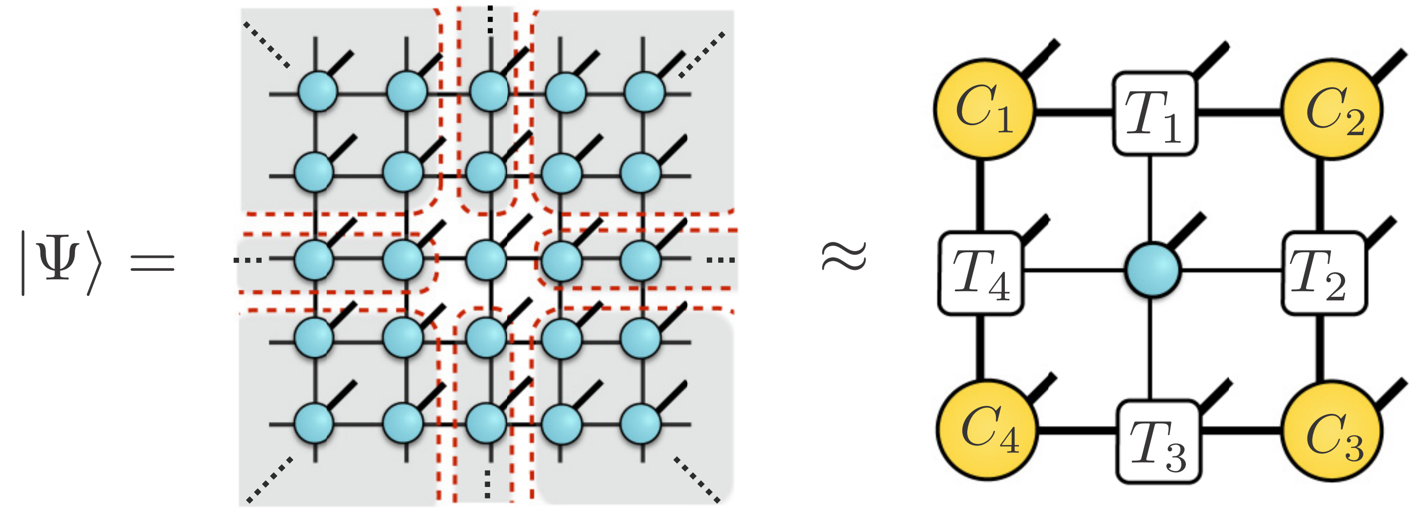




Quantum state renormalization scheme

Quantum state renormalization scheme

- The basic idea is to remove nonuniversal short-range entanglement related to the microscopic details of the system



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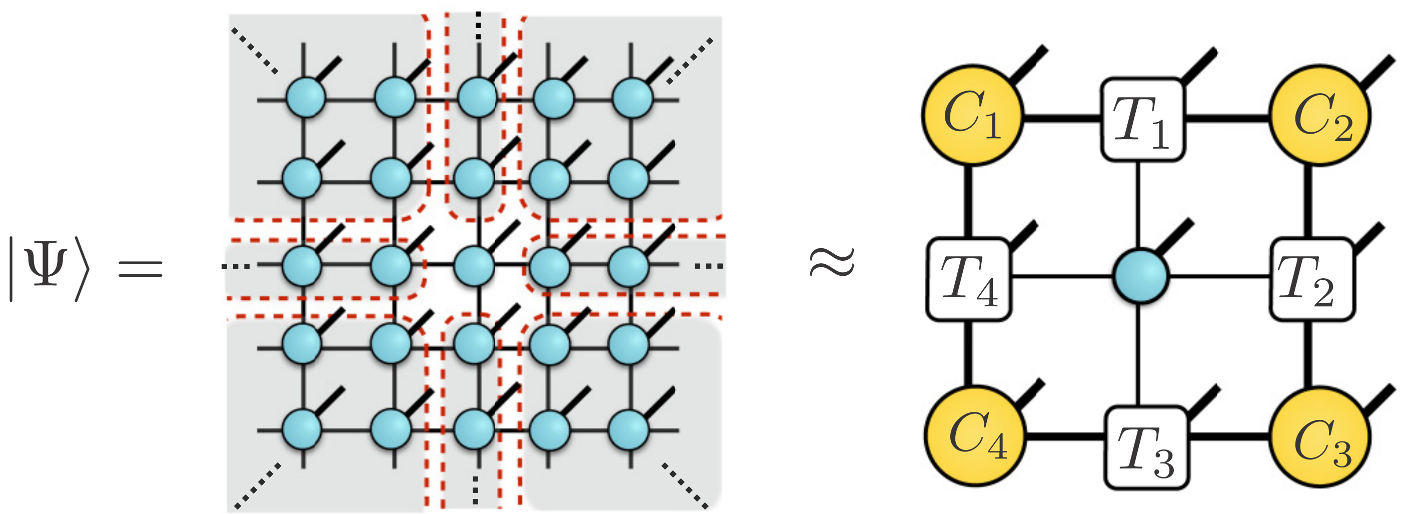
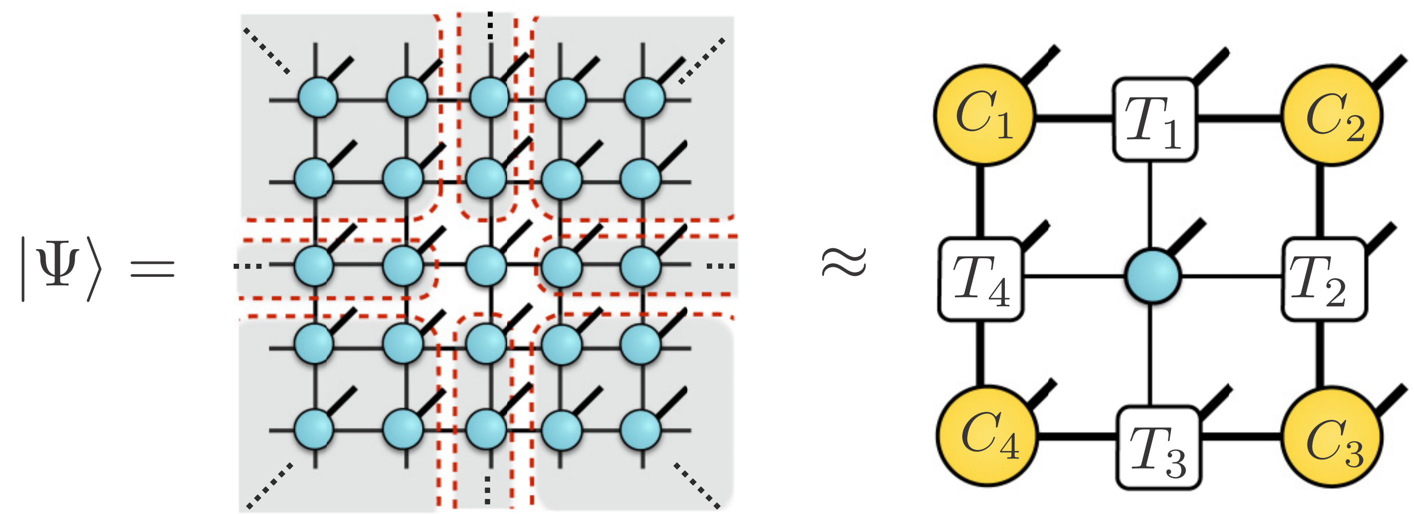
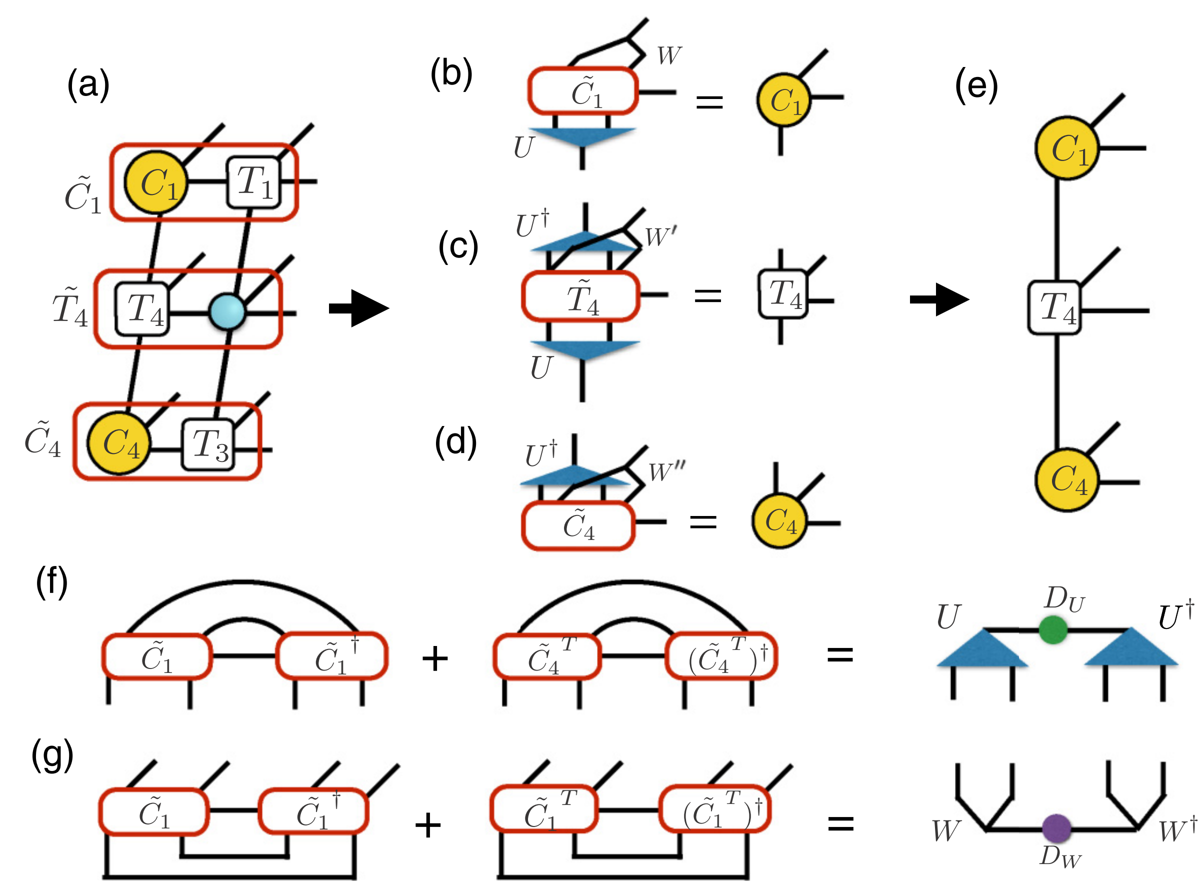


Quantum state renormalization scheme

- The basic idea is to remove nonuniversal short-range entanglement related to the microscopic details of the system
- The fixed-point wave function we make use of corner tensors
- The procedure is similar to the CTM approach but this time acting directly on the PEPS, which is single layer, and not on the TN for the norm, which is double layer.





Chiral topological corner entanglement spectrum

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- SU(2)k WZW chiral edge state is known to be critical and has a chiral gapless edge described by a SU(2) *Wess-Zumino-Witten (WZW) CFT.*

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- It can be represented by 2d *PEPS with D = 3* on a square lattice exactly.

$$A^{-1/2}_{2,0,1,1} = -\lambda_1 - i\lambda_2, \quad A^{-1/2}_{2,1,1,0} = -\lambda_1 + i\lambda_2,$$

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where $\lambda_0 = -2$, $\lambda_1 = 1$, and $\lambda_2 = 1$.

Chiral topological corner entanglement spectrum

- SU(2)k WZW chiral edge state is known to be critical and has a chiral gapless edge described by a SU(2) *Wess-Zumino-Witten (WZW) CFT.*
- It can be represented by 2d *PEPS with D = 3* on a square lattice exactly.
- The gapless edge state has been characterized by studying the *entanglement spectrum* of the PEPS on an infinitely-long but finite circumference cylinder

[D. Poilblanc, N. Schuch, and I. Affleck 2016
M. Mambrini, R. Orús, and D. Poilblanc 2016]

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$$A^{-1/2}_{0,1,2,1} = -\lambda_0;$$
$$A^{-1/2}_{1,2,0,1} = \lambda_1 + i\lambda_2, \quad A^{-1/2}_{0,2,1,1} = \lambda_1 - i\lambda_2, \quad A^{-1/2}_{1,2,1,0} = \lambda_0;$$
$$A^{-1/2}_{0,1,1,2} = \lambda_1 + i\lambda_2, \quad A^{-1/2}_{1,1,0,2} = \lambda_1 - i\lambda_2, \quad A^{-1/2}_{1,0,1,2} = \lambda_0;$$
$$A^{1/2}_{2,1,0,0} = \lambda_1 + i\lambda_2, \quad A^{1/2}_{2,0,0,1} = \lambda_1 - i\lambda_2, \quad A^{1/2}_{2,0,1,0} = \lambda_0;$$
$$A^{1/2}_{0,0,2,1} = \lambda_1 + i\lambda_2, \quad A^{1/2}_{0,1,2,0} = \lambda_1 - i\lambda_2, \quad A^{1/2}_{1,0,2,0} = \lambda_0;$$
$$A^{1/2}_{0,2,1,0} = -\lambda_1 - i\lambda_2, \quad A^{1/2}_{1,2,0,0} = -\lambda_1 + i\lambda_2,$$
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$$A^{1/2}_{0,1,0,2} = -\lambda_0,$$

where $\lambda_0 = -2$, $\lambda_1 = 1$, and $\lambda_2 = 1$.

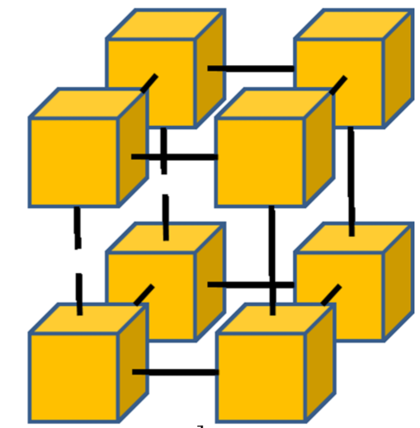
Chiral topological corner entanglement spectrum

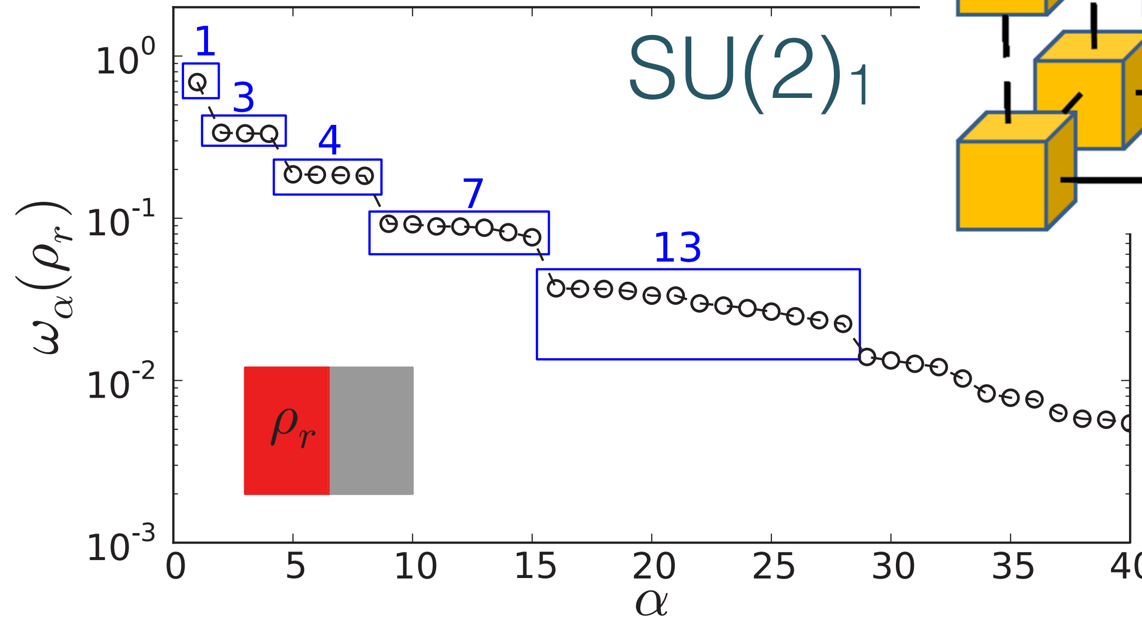
- $SU(2)_k$ WZW chiral edge state is known to be critical and has a chiral gapless edge described by a SU(2) *Wess-Zumino-Witten (WZW) CFT.*

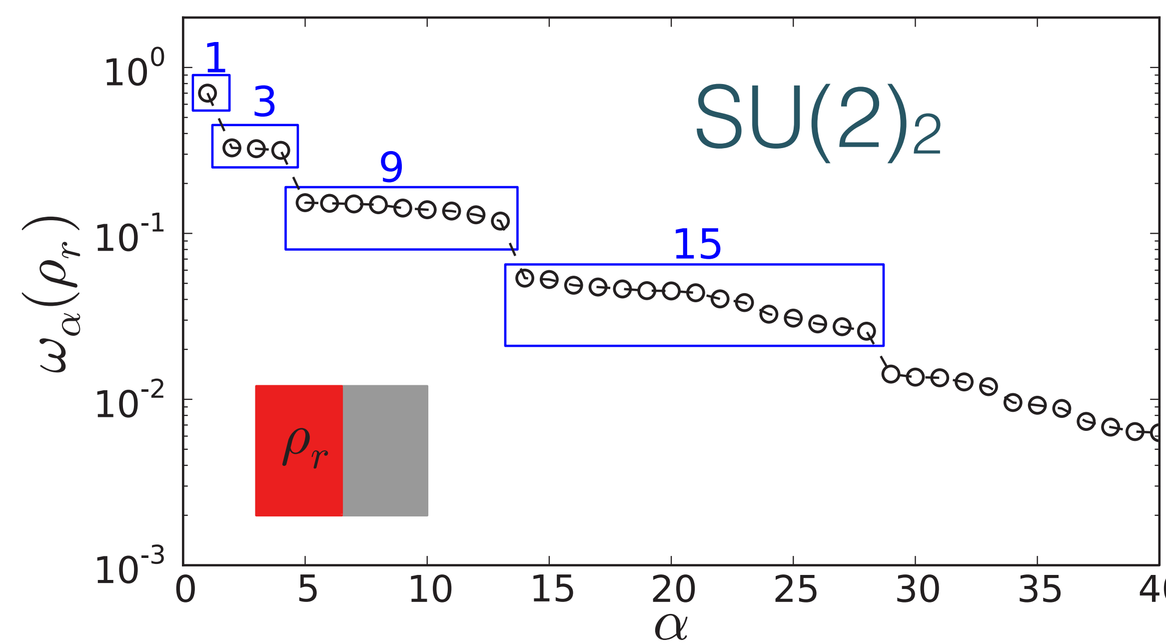
- It can be represented by 2d *PEPS with D = 3* on a square lattice exactly.

- The gapless edge state has been characterized by studying the *entanglement spectrum* of the PEPS on an infinitely-long but finite circumference cylinder

Using 2d QSRG



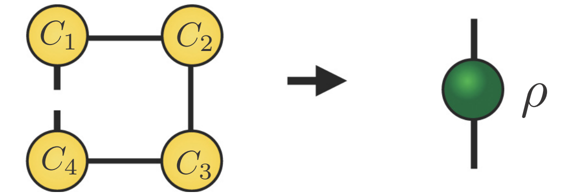


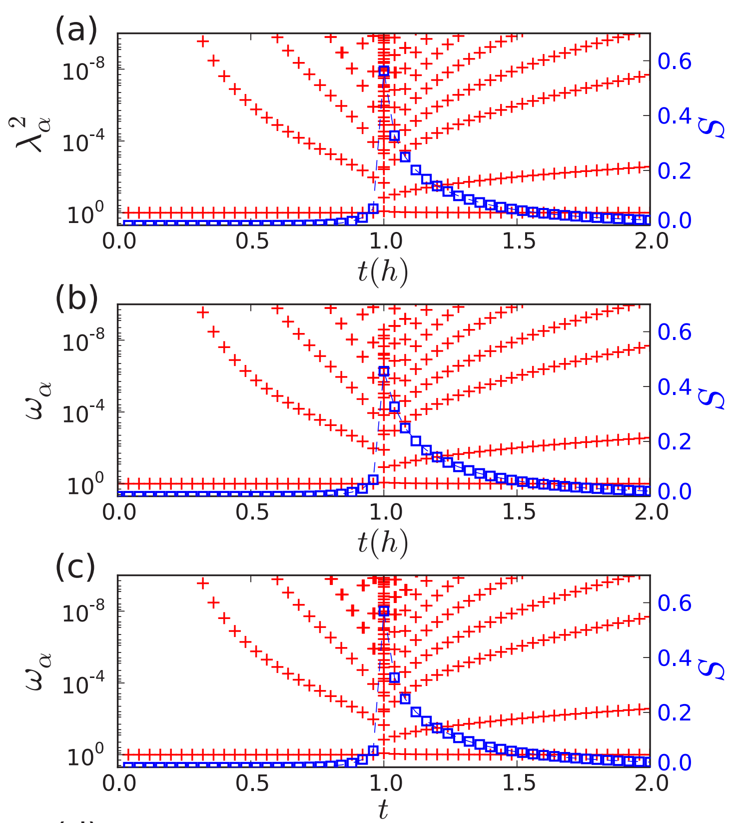


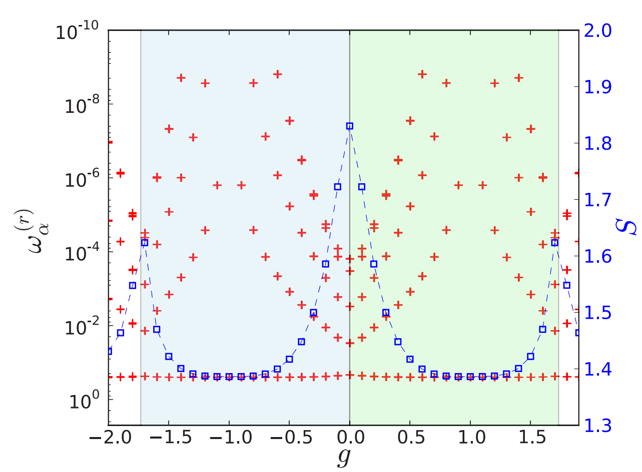
[D. Poilblanc, N. Schuch, and I. Affleck 2016
M. Mambrini, R. Orús, and D. Poilblanc 2016]

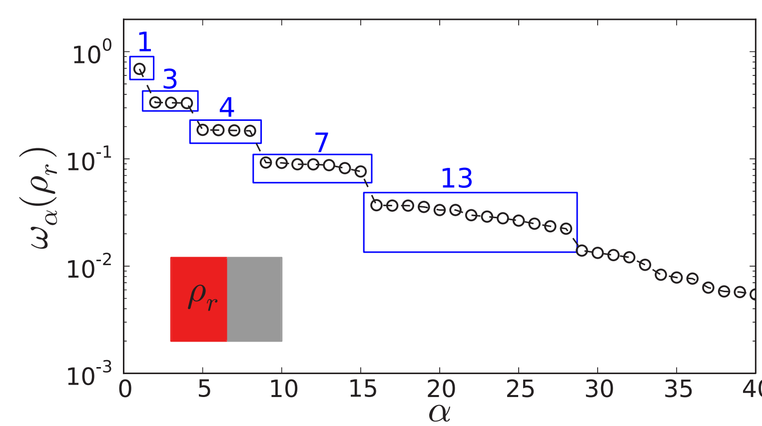
Conclusion

- We introduce the tensor network method and corner tensor
- We use corner properties to pinpoint quantum phase transitions
- We then consider chiral topological order phase









Thank you

Order parameter: From the wave function

- How to detect the topological order (TO) phase?

Quasiparticle excitations
with different
braiding statistics

Order parameter: From the wave function

- How to detect the topological order (TO) phase?

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braiding statistics

↓

Modular matrices

$$\langle\psi_a|\hat{S}|\psi_b\rangle = e^{-\alpha_S V + \mathfrak{o}(1/V)} S_{ab}$$

$$\langle\psi_a|\hat{T}|\psi_b\rangle = e^{-\alpha_T V + \mathfrak{o}(1/V)} T_{ab},$$

$\{|\psi_a\rangle\}_{a=1}^{N}$:degenerate ground state

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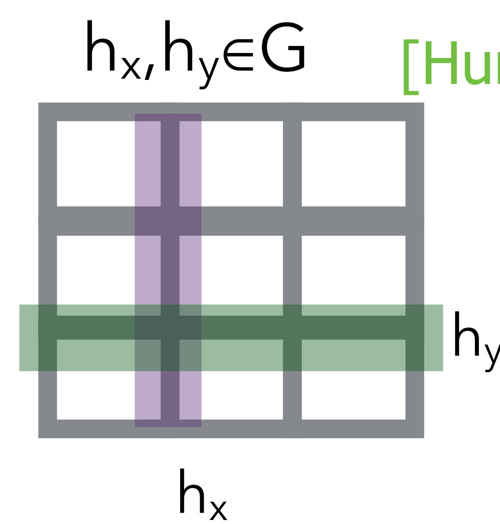
- Symmetry protected topological order (SPT)

Order parameter: From the wave function

- How to detect the topological order (TO) phase?
- Symmetry protected topological order (SPT)

Quasiparticle excitations with different braiding statistics

To use the symmetry twist to simulate degenerate ground state



[Hung & Wen 2014]

Modular matrices

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$$\langle \psi_a | \hat{T} | \psi_b \rangle = e^{-\alpha_T V + \mathfrak{o}(1/V)} T_{ab},$$

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Order parameter: From the wave function

- How to detect the topological order (TO) phase?

Quasiparticle excitations with different braiding statistics

Modular matrices

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$$\langle\psi_a|\hat{T}|\psi_b\rangle = e^{-\alpha_T V + \mathfrak{o}(1/V)} T_{ab},$$

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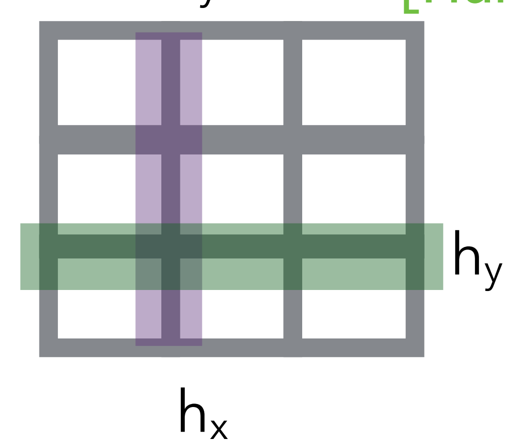
- Symmetry protected topological order (SPT)

To use the symmetry twist to simulate degenerate ground state

$h_x, h_y \in G$ [Hung & Wen 2014]



- Symmetry breaking phase

No degenerate ground state → Modular matrices is trivial