Conformal Properties of Instanton Solutions in Quantum Field Theories

Chuan-Tsung Chan
(Physics Division, NCTS)
傳 宗 詹

Outline of this talk :

I. The Use of Instanton

- A Quantum Mechanical Example (Double-Well Potential)

II. Instanton in Yang-Mills Gauge Theory

III. Symmetries & Instanton Measure

IV. Summary & Conclusion

The Role of Instanton in Quantum Physics

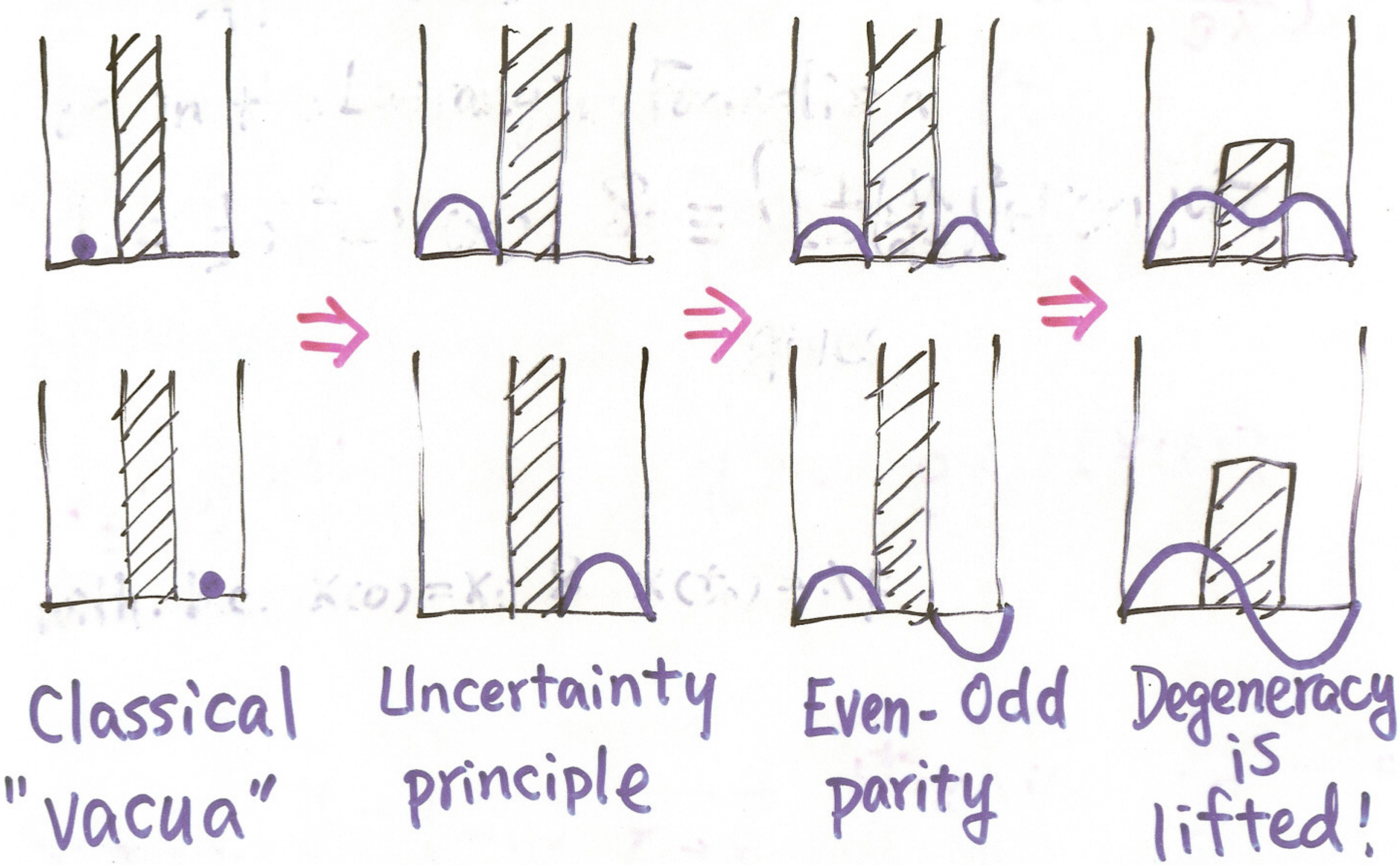
Instanton $\equiv$ Finite Action Solution to the Classical Euclidean Equation of Motion

$=$ Saddle-point of Euclidean Path Integrals

- typically carries topological charge.
- interpolates between classical vacua

Manifestation $\Rrightarrow$ TUNNELLING EFFECT!

Physical Picture of Tunnelling Effect



A More Realistic Model (Double-Well Potential)

$H = \frac{p^2}{2m} + V(x)$, where $V(x) \equiv \lambda\left(x^2 - \frac{m\omega^2}{8\lambda}\right)^2$

or in the Lagrangian Formalism $(\tau = it)$

$L = \frac{1}{2}\dot{x}^2 - V(x)$, $S_E \equiv \int \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] d\tau$

Feynman Path Integral gives

$$\langle x_f | e^{-iHt_0} | x_i \rangle = \mathcal{N} \int [\mathcal{D}x]\, e^{iS_M[x(t)]}$$

$\underline{iS_M = -S_E}$

with. b.c. $x(0) = x_i$ & $x(t_0) = x_f$

Feynman-Kac Formula (spectral information)

$$\langle x_f | e^{-H\tau_0} | x_i \rangle \xrightarrow[\tau_0 \to \infty]{} \psi_0^*(x_i)\, \psi_0(x_f)\, e^{-E_0 \tau_0}$$

The energy splitting $\Delta E = E_1 - E_0$ between $\psi_0(x)$ & $\psi_1(x)$ is a non-analytic function of λ.

To ~~extract~~ extract ΔE, one needs to compare the following Green's functions (propagators)

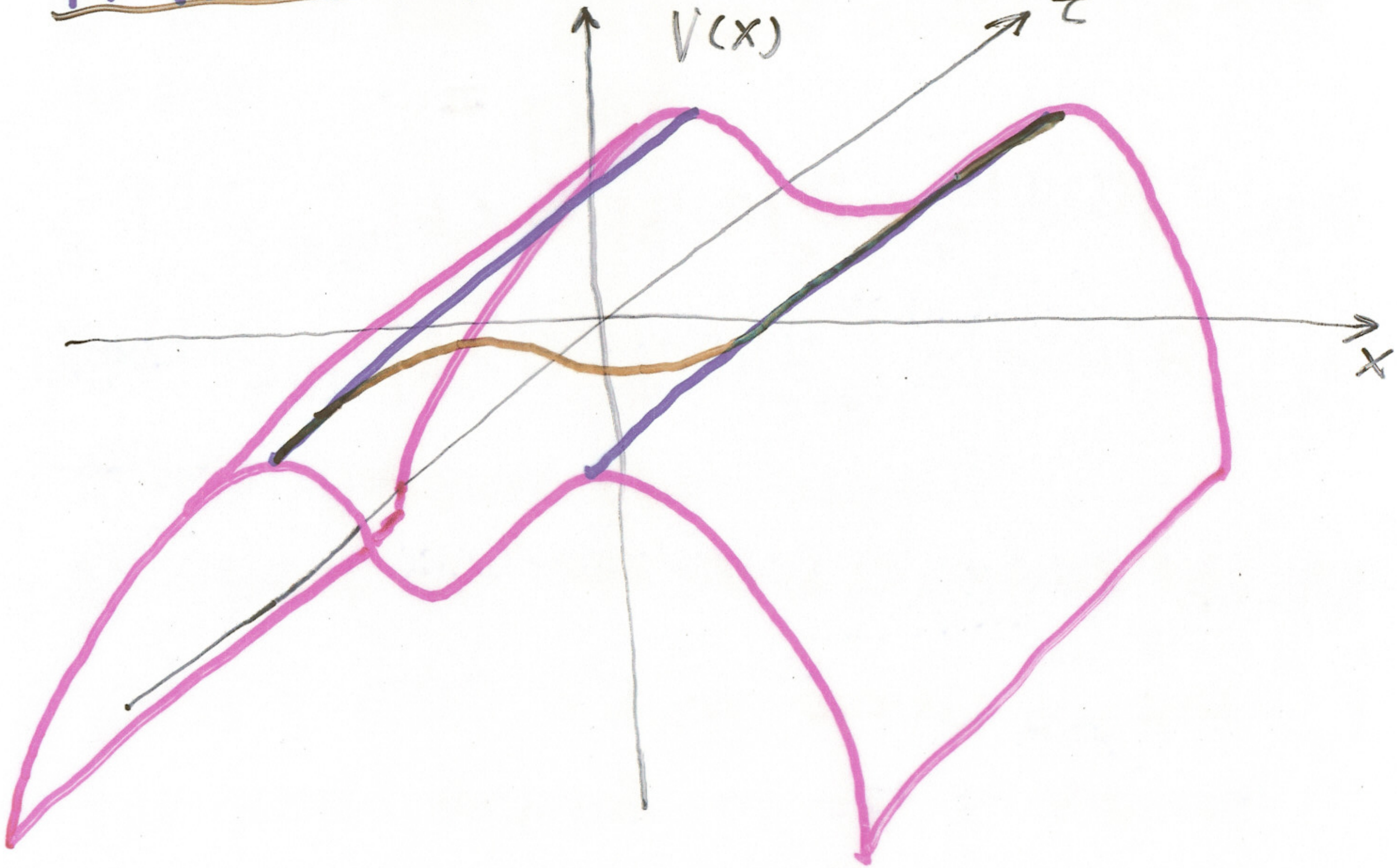
$$\langle -\eta | e^{-H\tau_0} | \eta \rangle \Rightarrow |\psi_0(\eta)|^2 e^{-E_0\tau_0} - |\psi_1(\eta)|^2 e^{-E_1\tau_0}$$

$$\langle \eta | e^{-H\tau_0} | \eta \rangle \Rightarrow |\psi_0(\eta)|^2 e^{-E_0\tau_0} + |\psi_1(\eta)|^2 e^{-E_1\tau_0}$$

in the $\tau_0 \to \infty$ limit

For the 1st case. the leading contribution to the path integral is given by one-instanton contribution.

Pictorial Representation of Instanton Dynamics



The leading contribution to the transition amplitude is given by

$$\langle -\eta | e^{-H\tau_0} | \eta \rangle \Rightarrow N \left[\det \left[-\frac{d^2}{d\tau^2} + V''(\bar{X}_c) \right] \right\}^{-\frac{1}{2}} e^{-S_0}$$

$$= N \left(\prod_n \omega_n^2 \right)^{-\frac{1}{2}} e^{-S_0}$$

But! Translation Invariance $\Rightarrow$ Existence of zero mode!

$$\frac{\delta S}{\delta X}\left(\bar{X}_c(\tau-\tau_c)\right) = 0 \Rightarrow \frac{d}{d\tau_c}\left[\frac{\delta S}{\delta X}\left(\bar{X}_c(\tau-\tau_c)\right)\right] = 0$$

$$\Rightarrow \left[\frac{\delta^2 S}{(\delta X)^2}\left(\bar{X}_c(\tau-\tau_c)\right)\right]\left(\frac{d}{d\tau_c}\bar{X}_c(\tau-\tau_c)\right) = 0$$

$$\left(= -\frac{d}{d\tau}\bar{X}_c(\tau-\tau_c)\right) \Rightarrow X_0(\tau)$$

Proper normalization

$$\Rightarrow \boxed{X_0(\tau) = \frac{1}{\sqrt{S_0}} \frac{d}{d\tau} \bar{X}(\tau-\tau_c)}$$

Semi-classical approximation of path integral

$$\langle -\eta | e^{-H\tau_0} | \eta \rangle = N \int_{\substack{x(-\frac{\tau_0}{2}) = \eta \\ x(\frac{\tau_0}{2}) - \eta}} \mathcal{D}x(\tau)\, e^{-S_E[x(\tau)]}$$

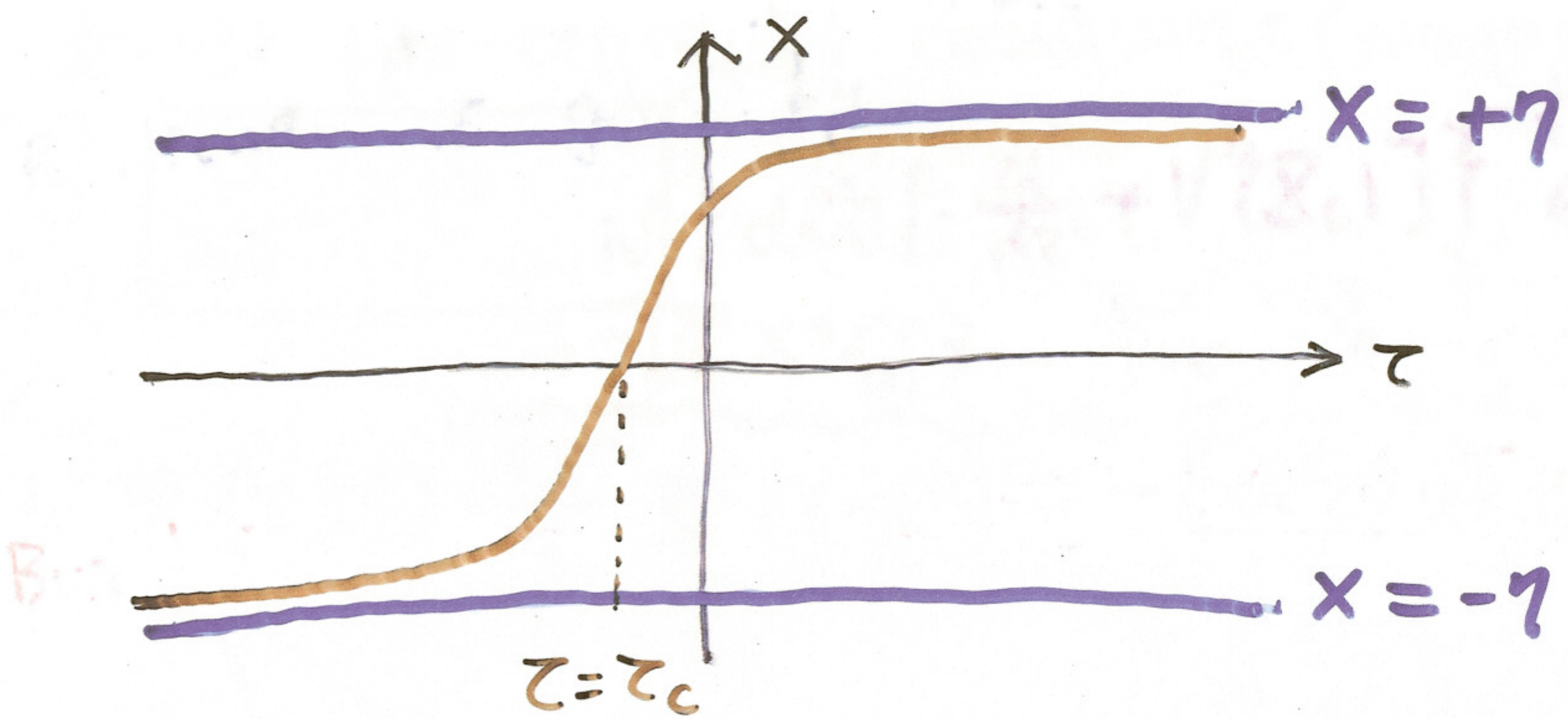
Expand the Euclidean Action around the one-instanton background $S_E[x(\tau)] = S_E[X_c(\tau)] + \frac{\delta^2 S_E}{(\delta x)^2} (\delta x)^2 + \cdots$

where $\delta x \equiv x(\tau) - X_c(\tau)$: fluctuation around the instanton background

$$= \sum_n C_n x_n \leftarrow \text{complete set}$$

$$\left[\frac{\delta^2 S_E}{(\delta x)^2} \right] x_n(\tau) = \underbrace{\omega_n^2}_{\text{eigenvalue}} x_n(\tau) \qquad \frac{\delta^2 S_E}{\delta x^2} = -\frac{d^2}{d\tau^2} + V''(X_c)$$

boundary conditions. $x_n(-\frac{\tau_0}{2}) = x_n(\frac{\tau_0}{2}) = 0$.

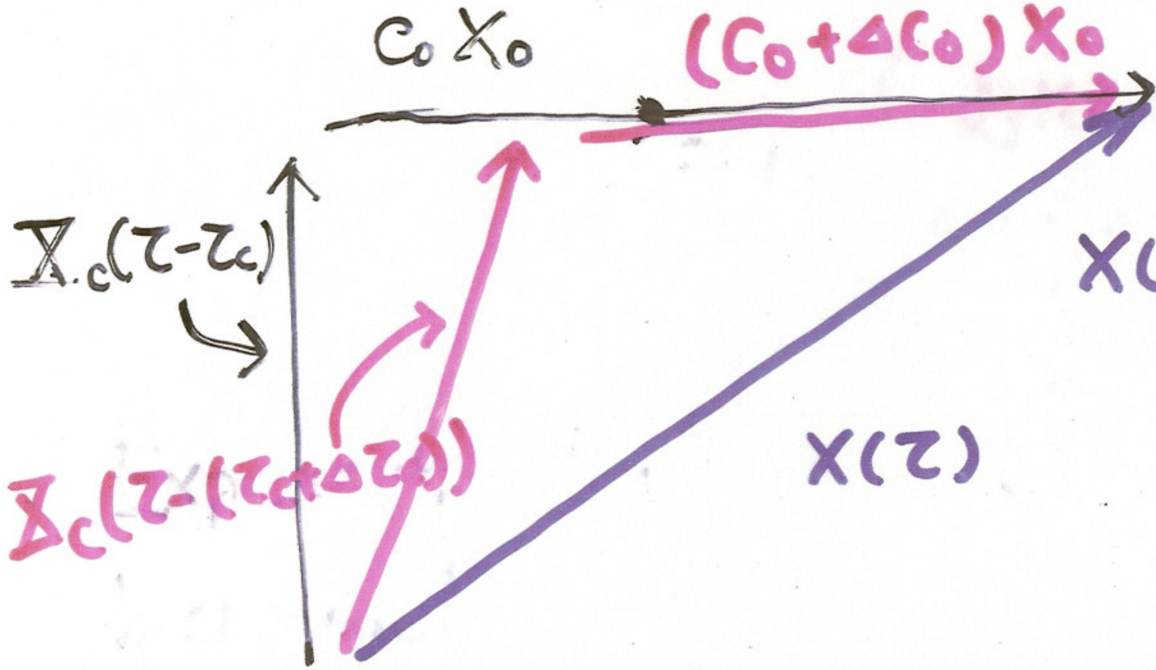


$$X_0(\tau) = \eta \tanh \frac{\omega(\tau - \tau_c)}{2} \quad , \quad \eta^2 = \frac{m\omega^2}{8\lambda}$$

with $S[X_0(\tau)] = \dfrac{\omega^3}{12\lambda} = \displaystyle\int_{-\infty}^{\infty} d\tau \, \dot{X}_0^2 = \int_{-\eta}^{\eta} \sqrt{2V}\, dx$

τ_0 is the position of instanton (in time), which is a free parameter corresponding to translational invariance of this model!!

Collective Coordinate and the associated Jacobian



$$X(\tau) = \mathcal{X}_c(\tau-\tau_c) + c_0(X_0(\tau))$$

$$= \mathcal{X}_c\big(\tau-(\tau_c+\Delta\tau_c)\big) + (c_0+\Delta c_0) X_0(\tau)$$

$$\Rightarrow (\Delta\tau_c) \underbrace{\frac{d}{d\tau_c}\mathcal{X}_c(\tau-\tau_c)}_{= -\sqrt{S_0}\, X_0(\tau)} = -(\Delta c_0) X_0(\tau)$$

$$\Rightarrow \boxed{\frac{dc}{d\tau_c} = \sqrt{S_0}} \rightarrow \text{Jacobian for change of variables.}$$

τ_c is the collective coordinate (moduli)!

Summation of the series

$$\langle -\eta | e^{-H\tau_0} | \eta \rangle = \sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}} \sinh(\omega\tau_0 d)$$

$$\langle \eta | e^{-H\tau_0} | \eta \rangle = \sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}} \cosh(\omega\tau_0 d)$$

where $d \equiv \sqrt{\frac{6}{\pi}} \sqrt{S_0}\, e^{-S_0}$ instanton density

$\Rightarrow$ spectral information

① $\Psi_0(\pm\eta) = |\Psi_1(\pm\eta)| \sim \left(\frac{\omega}{4\pi}\right)^{1/4}$

② $\Delta E = E_1 - E_0 = \sqrt{\frac{2\omega^3}{\pi\lambda}}\, e^{-\frac{\omega^3}{12\lambda}}\, \omega$

So $\det\left[-\frac{d^2}{d\tau^2}+V''(\bar{x}_0)\right] \ni \sqrt{\frac{S_0}{2\pi}}\, d\tau_c \underbrace{\left(\prod_{n\neq 0} \omega_n^2\right)}_{\text{non-zero modes}}$

- After some tedious algebra we can show that

$$\langle -\eta | e^{-H\tau_0} | \eta \rangle_{\text{one-inst}} = \left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right)\left(\sqrt{\frac{6}{\pi}}\sqrt{S_0}\, e^{-S_0}\right) \omega\, d\tau_c$$

- Inclusion of multi-instanton contribution (with dilute instanton gas approx.)

we get $\langle -\eta | e^{-H\tau_0} | \eta \rangle = \sum_{n=1,3..} \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2} \frac{(\omega\tau_0 d)^n}{n!}$

$$\langle \eta | e^{-H\tau_0} | \eta \rangle = \sum_{n=0,2} \sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}} \frac{(\omega\tau_0 d)^n}{n!}$$

Going to Yang-Mills Gauge Theory (SU(2))

$$S_E = \frac{1}{4}\int d^4x\, G^a_{\mu\nu} G^a_{\mu\nu}, \qquad G^a_{\mu\nu}\frac{\lambda^a}{2} = [D_\mu, D_\nu]$$

$$D_\mu = \left(\partial_\mu - igA^a_\mu \frac{\lambda^a}{2}\right)$$

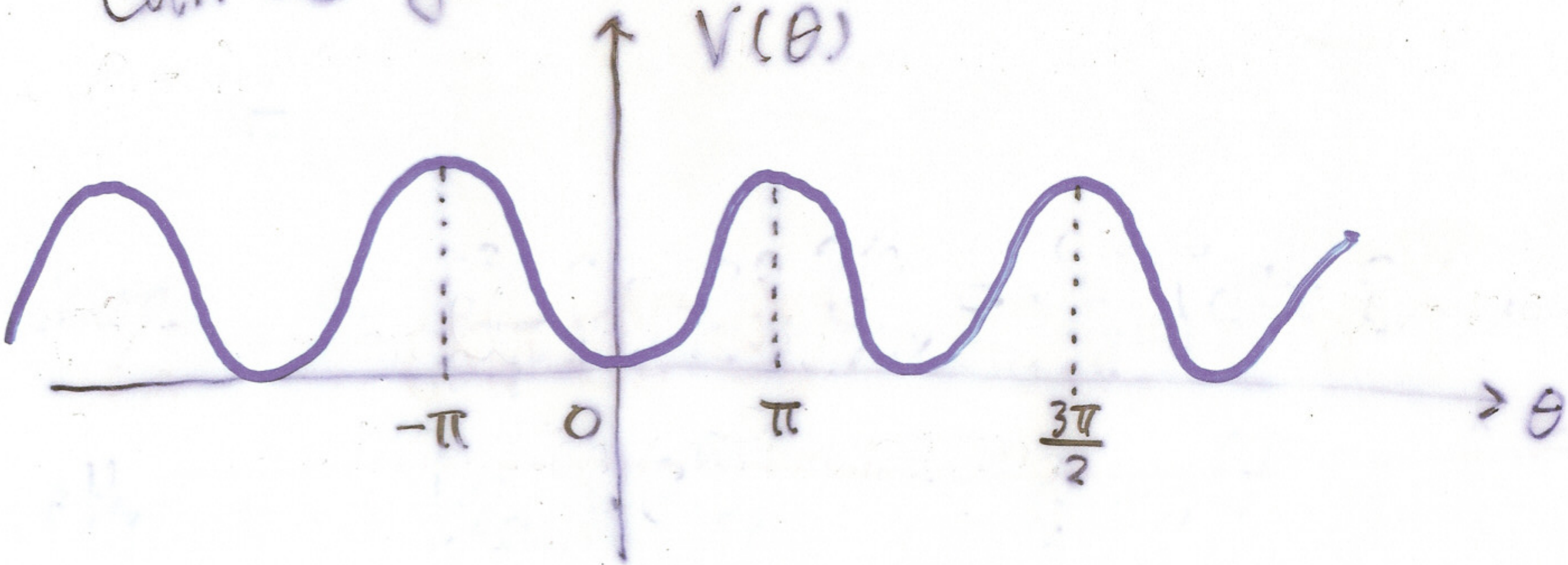
$$= \int d^4x \left[\pm\frac{1}{4} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \frac{1}{8}\left(G^a_{\mu\nu} \mp \tilde{G}^a_{\mu\nu}\right)^2\right]$$

$$= \pm\frac{8\pi^2}{g^2} Q + \frac{1}{8}\int d^4x \left(G^a_{\mu\nu} \mp \tilde{G}^a_{\mu\nu}\right)^2 \geq \pm\frac{8\pi^2}{g^2} Q$$

where $Q \equiv \frac{g^2}{32\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{g^2}{32\pi^2}\int d^4x\, \partial_\mu K_\mu$

with $\tilde{G}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} G^a_{\lambda\sigma}$, $K_\mu \equiv 2\varepsilon_{\mu\nu\lambda\sigma}\left(A^a_\nu \partial_\lambda A^a_\sigma + \frac{g}{3}\varepsilon^{abc} A^a_\nu A^b_\lambda A^c_\sigma\right)$

In a certain sense, there exists certain "tunnelling" effect in pure Yang-Mills theory. Roughly speaking, an analogus QM system can be given as follows



Ansatz: $A_\mu^a(x) = \frac{2}{g}\eta_{a\mu\nu} x_\nu \frac{f(x^2)}{x^2}$

proper boundary condition $f(x^2) \xrightarrow[|x|\to\infty]{} 1$, $f(x^2) \xrightarrow[|x|\to 0]{} \text{const.}\, x^2$

$$G_{\mu\nu}^a = -\frac{4}{g}\left\{\eta_{a\mu\nu}\frac{f(1-f)}{x^2} + \frac{x_\mu \eta_{a\nu\gamma} x_\gamma - x_\nu \eta_{a\mu\gamma} x_\gamma}{x^4}\left[(1-f)f - x^2 f'\right]\right\}$$

$$\tilde{G}_{\mu\nu}^a = -\frac{4}{g}\left\{\eta_{a\mu\nu} f' - \frac{x_\mu \eta_{a\nu\gamma} x_\gamma - x_\nu \eta_{a\mu\gamma} x_\gamma}{x^4}\left[(1-f)f - x^2 f'\right]\right\}$$

self-dual condition $\Rightarrow$ $f(1-f) = x^2 f'$

Solution $\Rightarrow$ $f(x^2) = \frac{x^2}{x^2+\rho^2}$

In general $A_\mu^a(x) = \frac{2}{g}\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}$

$$G_{\mu\nu}^a(x) = -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{[(x-x_0)^2+\rho^2]^2}$$

$G^a_{\mu\nu} = \tilde{G}^a_{\mu\nu}$ self-dual $\Rightarrow$ instanton

$G^a_{\mu\nu} = -\tilde{G}^a_{\mu\nu}$ anti-self-dual $\Rightarrow$ anti-instanton

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Explicit Form of BPST instanton with $Q=1$

$$g \frac{\tau^a}{2} A^a_\mu \xrightarrow[|x| \to \infty]{} i\, S_1 (\partial_\mu S_1^\dagger) \quad \text{pure gauge} \quad (SU(2))$$

If $S_1 = \dfrac{i \tau_\mu x_\mu}{\sqrt{|x|^2}} \Rightarrow A^a_\mu \xrightarrow[|x| \to \infty]{} \dfrac{2}{g} \eta_{a\mu\nu} \dfrac{x_\nu}{x^2}$

t'Hooft symbol

$\tau_\mu \equiv (\vec{\tau}, i)$

Pauli Matrices

To calculate the vacuum-vacuum transition amplitude we need to expand the vector potential A_μ^a around the instanton background $A_\mu^a = A_\mu^a(\text{ins}) + a_\mu^a$ & the Euclidean action to quadratic level.

$$S(A) = S_0 + \frac{1}{2}\int d^4x \, a_\mu^a \, L_{\mu\nu}^{ab}(A(\text{ins})) \, a_\nu^b$$

$$= \left(\frac{8\pi^2}{g^2}\right) - \frac{1}{2}\int d^4x \, a_\mu^a \left[D^2 a_\mu^a - D_\mu D_\nu a_\nu^a + 2g\epsilon^{abc} G_{\mu\nu}^b a_\nu^c\right]$$

NEW complications for evaluating path integral

① zero-modes due to gauge symmetry

⇒ Faddeev-Popov procedure

② UV divergence ⇒ Renormalization

One can identify moduli parameters

x_0 : center of the instanton (4)

ρ : size of the instanton (1) $\Rightarrow$ new

↳ due to scaling invariance

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① one can check the solution above gives $Q = 1$

② in the $A_0 = 0$ gauge, one can see that instanton field interpolate different vacuum configurations with $Q = k' - k$

In order to evaluate the instanton determinant with proper treatment of zero-modes.

$\Rightarrow$ Need to know the underlying symmetries!

At Classical level, for 4-dimensional Euclidean space, we expect the pure SU(2) Yang Mills gauge theory to enjoy ① conformal symmetry (15 generators)
② global color rotation (3 generators)

However, for the conformal part $15 = \underbrace{4}_{\text{translation}} + \underbrace{6}_{\text{rotation}} + \underbrace{4}_{\text{special conformal}} + \underbrace{1}_{\text{dilat.}}$

4 special conformal transformation can be thought as composition of translation and inversion

$$x_\mu \to x_\mu' = \frac{x_\mu}{x^2}, \quad A_\mu \to x'^2 A_\mu(x'^2)$$

$$\frac{2}{g}\eta_{a\mu\nu}\left(\frac{x_\nu}{x^2+1}\right) \xrightarrow{\text{inv}} \frac{2}{g}\eta_{a\mu\nu}\frac{x_\nu}{x^2(x^2+1)}$$

$$U = \frac{i\sigma_\mu x_\mu}{\sqrt{|x|^2}}$$

the resulting field lies at the same gauge orbit!

New Features for Quantum Field Theory

① gauge symmetry creates zero-modes!

Need to add an extra term (covariant gauge fixing)

$$\Delta S = \frac{1}{2}\int d^4x\,(D_\mu a_\mu^a)^2 = \frac{1}{2}\int d^4x\, a_\mu^a (\Delta L)_{\mu\nu}^{ab}\, a_\nu^b$$

$$\Delta S_{gh} = -\int d^4x\, \bar{\Phi}^a D^2 \Phi^a = \int d^4x\, \bar{\Phi}^a L_{gh}^{ab}\, \Phi^b$$

$$\Rightarrow \langle 0 | 0_T \rangle_{\text{one ins}} = [\det(L+\Delta L)]^{-\frac{1}{2}} (\det L_{gh})\, e^{-S_0}$$

where $|0_T\rangle \equiv e^{-HT}|0\rangle$, $S_0 = \frac{8\pi^2}{g^2}$

② Need Regularization

$$\Rightarrow \langle 0 | 0_T \rangle^{\text{Reg}}_{\text{one-ins}} \equiv \left[\frac{\det(L+\Delta L)}{\det(L+\Delta L+M^2)}\right]^{-\frac{1}{2}} \frac{\det L_{gh}}{\det(L_{gh}+M^2)}\, e^{-S_0}$$

Zero Modes & Instanton Measure

DWP lesson: For each zero mode, we need a factor of Jacobian $=\sqrt{S_0}$ together with an integral over the corresponding **collective coordinate (moduli)**

zero mode part in $[\det(L+\Delta L)]^{-\frac{1}{2}}[\det(L+\Delta L(A=0))]^{\frac{1}{2}}$

$$\Rightarrow \underbrace{\int d^4x_0}_{(4)}\, \underbrace{\rho^3 d\rho \sin\theta\, d\theta\, d\varphi\, d\psi}_{(1+3)}\ \underbrace{M^8}_{\text{regularization}} (\sqrt{S_0})^8$$

$$\Rightarrow \frac{\langle 0|O_T\rangle^{Reg}_{one\text{-}ins}}{\langle 0|O_T\rangle_{per.}} = \text{const.}\int \frac{d^4x_0\, d\rho}{\rho^5} \underbrace{\left(\frac{8\pi^2}{g_0^2}\right)^4}_{(\sqrt{S_0})^8} \exp\Big(\underbrace{-\frac{8\pi^2}{g_0^2}}_{e^{-S_0}} + \underbrace{8\ln M\rho}_{M^8} + \underbrace{\Phi_1}_{\text{non-zero mode}}\Big)$$

Based on spinorial formalism, one can show that, while we have 6 Euclidean Rotations + 3 color rotation (for $SU(2)$ gauge group) at hands. only 3 linear combinations of them act on the one-instanton solution non-trivially, resulting 3 extra collective coordinates.

Thus, overall, we have $4+1+3=8$ zero modes for one-instanton configuration.

The Infrared Problem of Large instantons

$$\frac{\langle 0|O_T\rangle^{Reg}_{one\text{-}ins}}{\langle 0|O_T\rangle_{per}} \propto \int_0^\infty \frac{d\rho}{\rho^5}\,\frac{\exp\left[-\frac{8\pi^2}{\bar{g}(\rho M)^2}\right]}{[\bar{g}(\rho M)]^8}$$

For weak coupling $\bar{g} \ll 1$, the running coupling constant is obtained by solving the RG equation

$$\frac{d\bar{g}(\rho M)}{d[\ln(\rho M)]} = \frac{11}{24\pi^2}[\bar{g}(\rho M)]^3 + O(\bar{g}^5)$$

$$\Rightarrow \left[\frac{1}{\bar{g}(\rho M)}\right]^2 = \frac{1}{g_0^2} - \frac{11}{12\pi^2}\ln(\rho M)$$

UV divergent!

$$\Rightarrow \text{Integrand } \frac{e^{-\frac{8\pi^2}{g_0^2}}}{g_0^8}\,\frac{(\rho M)^{22/3}}{\rho^5}\left[1 - \frac{11 g_0^2}{12\pi^2}\ln(\rho M)\right]^4$$

Two Ways out $\Rightarrow$

① Introduce UV cut-off via Higgs Mechanism $\Rightarrow \rho_{Max} \sim \frac{1}{M_W}$

② Invoke Supersymmetry

$\Rightarrow$ Exact β-function