Dynamics of Spacetime from a Large N Matrix Model

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Yang-Mills (YM) type: -tr[A,A][A,A]/4

1. Introduction and Motivation

String Theory

- ***Fundamental degrees of freedom are extended object, not point like particles***

- ***A nice candidate for the unified quantum theory with gravity***

- ***To investigate strong gravity phenomenon (Black Hole, very early Universe), 2nd quantized formulation may be needed.***

Main remarkable properties from this extension:

1. Infinitely many state of string vibration

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Infinitely many one particle state
(include graviton, gauge boson, …)

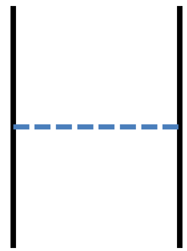
2. Formulated in higher dimension
(e. g. D=26, D=10 flat space)

3. UV finiteness of loop amplitude

String theory is a nice candidate of the unified quantum theory including gravity.

- ***Well established formulation is 1-st quantized/ perturbative formulation***

Coupling constant of gravitational correction in perturbation theory (D=10):



$$(E^8 G_N)$$

G_N: Newton constant in 10D

Perturbation theory will breaks down in processes with very high energy.

There will be large back reaction to spacetime if

$$E > \frac{1}{g^{1/4}\sqrt{\alpha'}}$$

g: string coupling. α': (string length scale)^2.

There could be black hole production
with Schwatzschild radius:

$$R \sim (G_N E)^{1/7}$$

- *To investigate strong gravity phenomenon (Black Hole, very early Universe), 2nd quantized formulation may be needed.*

Proposals for 2nd quantized theory

- ***String Field Theory***

  Reproduces perturbative expansion of S-matrix

  Still difficult to see non-perturbative effect

  (Recent Progresses :Tachyon condensation, D-bane solution, ….   )

✓ - ***Reformulation by Large N Matrix Theory***

  Proposals for critical superstring in recent 15 years

  1 dim model (BFSS), 0 dim model (IKKT), etc.

Matrix Theory proposal (I) : IKKT model

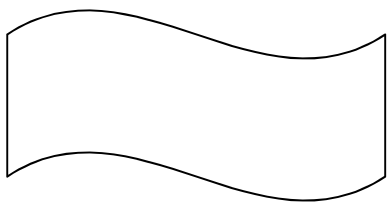
Ishibashi, Kawai, Kitazawa, Tsuchiya (1996)

Fundamental degrees of freedom:
N × N hermitian matrices $A_{(a)}$, ψ

$$S = -\frac{1}{4g^2} tr\left([A_a, A_b][A^a, A^b]\right) + \frac{1}{2g^2} tr\left(\bar{\psi}\Gamma^a[A_a, \psi]\right)$$

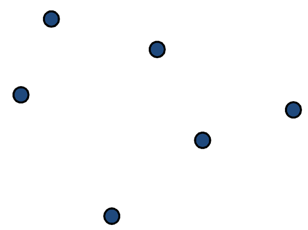
Large N

Multi - String



String World Sheet

distribution of diagonal elements



Large N

space-time picture

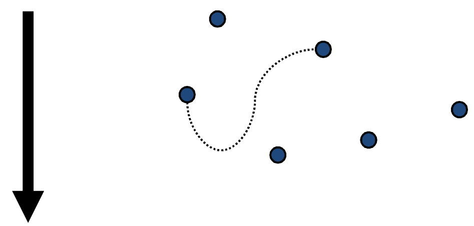
Matrix Theory proposal (II) : BFSS model

Banks, Fishler, Shenker, Susskind (1996)

Large N matrix quantum mechanics $X^{(I)}(t), A_{(a)}(t), \Psi(t)$

$$S_{BFSS} = \int dt\, tr\left(-\frac{1}{2}[D_0, X_{(i)}][D^0, X^{(i)}] + \frac{1}{4}[X_{(i)}, X_{(j)}][X^{(i)}, X^{(j)}] - \frac{1}{2}\bar{\Psi}D_0\Psi - \frac{1}{2}\bar{\Psi}\Gamma^{(i)}[X^{(i)}, \Psi]\right)$$

D0 brane effective theory in type IIA superstring



Large N limit formulates M-theory on a circle in light-cone direction

Having these Yang-Mills matrix models, we have a possibility to study the dynamics of spacetime that emerges from matrix.

→ *"Quantum theory of gravity"*

Purpose of this talk:

To introduce a part of studies that have been done along this interest.

Contents

Matrix Description of space(-time)

Space time may be described by matrix degrees of freedom

Dynamics of diagonal element
--> similar to the statistical system of polymer

There are toy examples that is more similar to ordinary space.

- *Two Examples*

  1. *Fuzzy Sphere*
  2. *Fuzzy Torus*

Matrix Description of space(-time) I: Fuzzy Sphere

Spin 1/2 operators are gives by 2 × 2 matrices (Pauli matrices)

$$J_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

Eigen states: | l , m >

$$J_3 |l, m> = m |l, m>$$
$$J^2 |l, m> = l(l+1) |l, m>, \quad J^2 = J_1^2 + J_2^2 + J_3^2$$

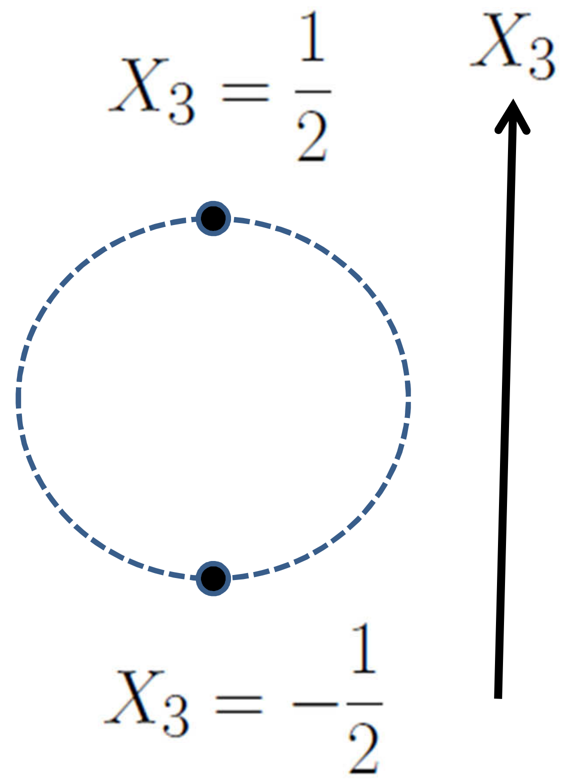
For 2 × 2 matrices, l=1/2 and m= 1/2, -1/2

Matrices J_a can be identified with coordinate X_a (operator!)

| l , m > :

Eigenstate of coordinate operators

$$X_1^2 + X_2^2 + X_3^2 = \frac{3}{4}$$



This is a sphere with only 2-point.

$$[X_a, X_b] = i\epsilon_{abc}X_c$$

Coordinates are **non-comutative** → " Fuzzy"

Field on Fuzzy Sphere

Hermitian Matrix

$$A = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2aY_{0,0} + \sqrt{2}(b-ic)Y_{1,-1} + \sqrt{2}(b+ic)Y_{1,1} + 2dY_{1,0}$$

$Y_{l,m}$: ***Matrix spherical harmonics***

Eigen matrix of

$$[J_3, Y_{l,m}] = mY_{l,m},$$

$$[J_1,[J_1,Y_{l,m}]] + [J_2,[J_2,Y_{l,m}]] + [J_3,[J_3,Y_{l,m}]] = l(l+1)Y_{l,m}.$$

$\longleftrightarrow$ $\phi(\theta,\phi) = \sum_{l,m} \phi_{l,m} Y_{l,m}(\theta,\phi)$ ***on sphere.***

2 ×2 matrix → N ×N matrix

(N = 2l+1)

$$(Y_{p,m})_{s,t} = (-1)^{l-s}\sqrt{2p+1}\begin{pmatrix} l & p & l \\ -s & m & t \end{pmatrix}$$

3j-symbol

p = 0,1, …., 2l

m = -p, -p+1, …, -1, 0, 1, … p-1, p

N ×N matrix → ∞ × ∞ matrix (N→∞)

Continuum limit

Matrix description of space(time) II: Fuzzy Torus

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

property: $U^2 = V^2 = \mathbf{1}, \qquad UV = (-1)VU$

Hermitian matrix

$$A = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} = a\mathbf{1} + dU + bV + icUV.$$

$\longleftrightarrow \quad \phi(\sigma_1, \sigma_2) = \sum_{m,n=0,1} \phi_{m,n}\, e^{im\sigma_1 + in\sigma_2}$

$$U = e^{i\hat{\sigma}_1}, \qquad V = e^{i\hat{\sigma}_2}$$

$$[\hat{\sigma}_1, \hat{\sigma}_2] = -i\pi$$ ***: non-commutativity***

Derivative:

$$[\hat{\sigma}_1, \ \ ] \Leftrightarrow -i\pi \frac{\partial}{\partial \sigma_2}, \qquad [\hat{\sigma}_2, \ \ ] \Leftrightarrow -i\pi \frac{\partial}{\partial \sigma_1}.$$

Kinetic term:

$$\frac{1}{\pi^2} \sum_{i=1,2} [\sigma_i, \phi][\sigma_i, \phi] \Leftrightarrow \sum_{i=1,2} \partial_i \phi \partial_i \phi$$

2 ×2 matrix → N ×N matrix

$$U_{ij} = \omega^{i-1}\delta_{ij}, \quad (\omega = e^{\frac{2\pi i}{N}}),$$
$$V_{ij} = \delta_{i,j-1},$$

$$U^N = 1, \quad V^N = 1, \quad UV = \omega^{-1}VU,$$

Non-commutativity:

$$[\hat{\sigma}_1, \hat{\sigma}_2] = -\frac{2\pi i}{N}$$

Matrix Model and Fuzzy Space (Dynamics)

Decompose matrix A as A=X+a

X: back ground , a: field

Yang-Mills matrix model → Gauge theory on X

$$-\frac{1}{4}Tr[A_i, A_j]^2$$

derivative

$$[A_1, A_2] = [X_1, X_2] + [X_1, a_1] - [X_2, a_1] + [a_1, a_2].$$

⇕

$$(\; F_{12} = \partial_1 a_2 - \partial_2 a_1 + [a_1, a_2] \;)$$

Quantum Correction

partition function: $$Z = \int \mathcal{D}A \, e^{-S[A]}$$

(A=X+a, integrate over a) $$= \int \mathcal{D}X \, e^{-\Gamma[X]}$$

Integration of X ≈ Taking saddle point:

$\delta\Gamma[X]/\delta X=0$

Quantum correction of fuzzy space X can be calculated in this way.

Stability Analysis

- *fuzzy S2 (2d)* — Imai, Kitazawa, Takayama, D.T [03]
- *fuzzy S2 ×S2 (4d)* — Imai, Kitazawa, Takayama, D.T [03]; Imai, Takayama [03]
- *fuzzy S2 ×S2 ×S2 (6d)* — Kaneko, Kitazawa, D.T [05]
- *fuzzy spaces with SU(3) isometry (4d, 6d)* — Kaneko, Kitazawa, , D.T [05]

4d fuzzy spaces are more stable than 2d, 6d fuzzy spaces.

More on the matrix description of space-time

We saw

Matrix can describe

Non-commutative sphere, torus

Fields on these spaces

More general curved spacetime ?

(Black Hole, RW spacetime, etc.)

Einstein Equation?

Approach by **Hanada, Kawai, Kimura (HKK) [05]**

Matrix with ∞ size → *Differential operator in curved spacetime*

$$A_{(a)} = R_{(a)}{}^{a}(g) D_a(\nabla, \omega)$$

∇: covariant derivative

ω: spin connection

Example: Covariant derivatives on 2-sphere

Two sphere (with stereographic coordinate)

$$\nabla^{[z]}_{(+)} = e^{-2i\theta}\left((1+z\bar{z})\partial_z + \frac{i}{2}\bar{z}\partial_\theta\right),$$
$$\nabla^{[z]}_{(-)} = e^{2i\theta}\left((1+z\bar{z})\partial_{\bar{z}} - \frac{i}{2}z\partial_\theta\right).$$

around the north pole of sphere.

Description include south pole is achieved by

$z = 1/w.$ *and* $\theta' = \theta + \arg(z) + \frac{\pi}{2}.$ *Then*

$$\nabla^{[w]}_{(+)} = e^{-2i\theta'}\left((1+w\bar{w})\partial_w + \frac{i}{2}\bar{w}\partial_{\theta'}\right),$$
$$\nabla^{[w]}_{(-)} = e^{2i\theta'}\left((1+w\bar{w})\partial_{\bar{w}} - \frac{i}{2}w\partial_{\theta'}\right),$$

Generally , matrix may be expanded as like

local Lorentz generator

$$A_{(a)} = a^{(0)}_{(a)} + a^{(1)}_{(a)}{}^{(b)} i\nabla_{(b)} + b^{(1)}_{(a)}{}^{bc} \mathcal{O}_{bc} + a^{(2)}_{(a)}{}^{(b)(c)} i^2 \nabla_{(b)} \nabla_{(c)} + \cdots$$

gauge field *vielbein* *spin connection* *higher spin fields*

There are Infinite number of fields with various spin.

Applications of the HKK : "A $\sim \nabla$ "

Driving

1. Classical gravity equation

2. Quantum gravity equation

from the YM- matrix model : S= -Tr [A,A][A,A]/4

Classical Gravity Equations from the Matrix Model

Matrix equation of motion:

$$[A^{(b)}, [A_{(b)}, A_{(a)}]] = 0$$

provides equations for infinitely many fields with various spins .

→ ***truncate only*** $A \to \nabla_a = e_a{}^b \partial_b + \omega_a^{bc} \mathcal{O}_{bc}$

vielbein: $e_\mu{}^a$, spin connection: $\omega_a{}^{bc}$

Gravity Equations

(Furuta, Hanada, Kawai, Kimura [06])

$$R_{ab} + \nabla^c T_{b,ca} + T^{p,q}{}_a T_{b,pq} = 0,$$

$$\nabla^a R_{abcd} + T^{p,q}{}_b R_{pqcd} = 0,$$

torsion: $$-T^c{}_{,ab} = e_\mu{}^c(\partial_a e^\mu{}_b - \partial_b e^\mu{}_a) + \omega_{ab}{}^c - \omega_{ba}{}^c,$$

curvature: $$R_{ab}{}^{cd} = \partial_a \omega_b{}^{cd} - \partial_b \omega_c{}^{cd} + \omega_a{}^c{}_e \omega_b{}^d{}_e - \omega_b{}^c{}_e \omega_a{}^d{}_e$$

Including torsion degrees of freedom

Classical solutions **(H. Isono, D.T [09])**

Time-dependent solutions with torsion:

- *FRW type metric ansatz*
- *Expansion with acceleration*

  → *application to dark energy?*

Spherical symmetric solution with torsion

- *Black hole solutions*
- *All founded solutions are*

  asymptotically non-flat

Quantum Gravity Equation from Matrix Model

Schrodinger equation of <u>matrix quantum mechanics</u>:

$$i\frac{\partial}{\partial t}\Psi(A,t) = H(A,\frac{\partial}{\partial A})\Psi(A,t)$$

After HKK : "A ∼ ∇ ",

$$i\frac{\partial}{\partial t}\Psi(\nabla,t) = H(\nabla,\frac{\partial}{\partial \nabla})\Psi(\nabla,t)$$

We get a quantum gravity equation.

Explicit form

$$i\frac{\partial}{\partial t}\Psi = \int d^dx\sqrt{h_d}\; tr_{\hat{g}}\Big\{ \frac{1}{\sqrt{h_d}}\frac{1}{2}(\frac{\delta}{\delta e_i^K}\partial_K)\sqrt{h_d}(\frac{\delta}{\delta e_i^L}\partial_L) + \frac{1}{2}(\frac{\delta}{\delta \omega_i{}^{jk}}\mathcal{O}_{jk})(\frac{\delta}{\delta \omega_i{}^{lm}}\mathcal{O}_{lm}) - \frac{1}{\sqrt{h_d}}\frac{1}{4}(T_{ij}{}^K\partial_K)\sqrt{h_d}(T_{ij}{}^L\partial_L) - \frac{1}{4}(R_{ij}{}^{kl}\mathcal{O}_{kl})(R_{ij}{}^{mn}\mathcal{O}_{mn})\Big\}\Psi.$$

(T. Matsuo, W-Y. Wen, S. Zeze, D.T [08])

Covariant only in spatial directions, no boost invariance

metric: $-dt^2 + h_d(t,x)_{IJ}dx^I dx^J$

Divergent quantities appear. Some regularization is required.

Minisuperspace Model

$$ds^2 = -dt^2 + \frac{1}{y(t)^2}dx^2.$$

Quantum mechanics of y(t)

Schrodinger equation was explicitly solved.

A wave packet solution was constructed

$$\Psi(t,y) = \sqrt{\frac{(2\beta)^{\frac{5}{3}}}{\Gamma(5/3)}}\frac{y^2}{3^{\frac{7}{6}}(\beta+iAt)^{\frac{5}{3}}}\exp\left[-\frac{y^3}{9(\beta+iAt)}\right].$$

This wave function describes a shrinking universe in $|t| \to \infty$

$$\langle\, g_{xx} \,\rangle = \langle\, y^{-2} \,\rangle = \frac{1}{\Gamma(\frac{5}{3})}\left(\frac{2}{9}\right)^{\frac{2}{3}}\left[\frac{\beta}{\beta^2+A^2t^2}\right]^{\frac{2}{3}}.$$

Number of questions have been left ………

- *Full covariance? ----- IKKT model may give a covariant equation. But to do this, we have to derive a sort of the Schrodinger equation from the model without time.*

- *What is a good regularization? ----- by introducing finite size matrix effect?, imposing some gauge condition?*

- *How go proceed further beyond the mini-superspace model ?*

Summary and Outlook

Various spacetime dynamics are studied by IKKT matrix model

Finite size matrix : *Non-commutative space*

Well defined, Gravity is not-manifest.

Stability analysis through the effective action

Infinite size matrix: *HKK prescription* $A \sim \nabla$

More formal, Gravity is manifest.

We got quantum and classical equations of a torsion gravity

We need more understanding on large N (matrix size → ∞) limit......

Matrix Renormalization Group (matrix RG) *may be a useful approach.*

Integrating out a part of matrix → flow equation

Links to the critical phenomena, Wilsonian RG are also expected.

(END)