Lecture IV: Residue Calculus

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This lecture introduce the residue calculus.

I. THE RESIDUE THEOREM

Definition of Residue:

If $f(z)$ is analytic in a neighborhood of $z = a$ but not at a, the residue of $f(z)$ at $z = a$ is $\frac{1}{2\pi i}\oint_C f(z)dz$ where C is the curve encloses a. It is denoted as $Resf(a)$.

From Cauchy Formula, the residue of $f(z)$ is identical to the coefficient c_{-1} of its Laurent series.

Example 4.1:

Please show that for a pole of order m, $Resf(a)$=$\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$.

Solution:

Since $z = a$ is the order m pole thus

$$f(z) = \sum_{n=-m}^{n=\infty} c_n (z-a)^n .$$

Therefore $g(z) = (z-a)^m f(z)$ is analytic at $z = a$ and its neighborhood. Apply Cauchy formula

$$\oint f(z)dz = \oint \frac{g(z)}{(z-a)^m}dz = \frac{2\pi i}{(m-1)!}\frac{d^{m-1}g(z)}{dz^{m-1}} = \frac{2\pi i}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right].$$

Remember $Resf(a)$=$\frac{1}{2\pi i}\oint_C f(z)dz$, so that

$$Resf(a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right].$$

Example 4.2:

Find the residue of $\tan z$ at $z = \frac{\pi}{2}$.

$$\begin{aligned}
\tan z &= \frac{\sin\left(z-\frac{\pi}{2}+\frac{\pi}{2}\right)}{\cos\left(z-\frac{\pi}{2}+\frac{\pi}{2}\right)} = \frac{\cos\left(z-\frac{\pi}{2}\right)}{-\sin\left(z-\frac{\pi}{2}\right)} = \frac{1}{z-\frac{\pi}{2}}\cos\left(z-\frac{\pi}{2}\right)\cdot\left(\frac{z-\frac{\pi}{2}}{\sin\left(z-\frac{\pi}{2}\right)}\right) \\
&= \frac{1}{z-\frac{\pi}{2}}\left[1-\frac{(z-\pi/2)^2}{2!}+...\right]\left[\frac{1}{1-\frac{(z-\pi/2)^2}{3!}+...}\right] \\
&= \frac{1}{z-\frac{\pi}{2}}\left[1-\frac{(z-\pi/2)^2}{2!}+...\right]\left[1+\frac{(z-\pi/2)^2}{3!}+...\right] \\
&= \frac{1}{z-\frac{\pi}{2}}\left[1-\frac{1}{6}(z-\pi/2)^2+...\right] = \frac{1}{z-\frac{\pi}{2}}+...
\end{aligned}$$

The residue is 1.

Residue Theorem:

If a function is analytic in a simply connected domain D except for a finite number of isolated singularities. Curve C is within D, then

$$\oint_C f(z)dz = 2\pi i\sum_{i=1}^{N} Resf(z_n).$$

here z_n are the singularities of $f(z)$ within C.

Home Work 4.1:

(a) Find out the $Res(z=1)$ for $f(z)=\frac{e^{tz}}{(z+2)(z-1)^4}$.

(b) Find out the poles and the associated residues of $f(z)=\frac{z^2e^z}{1+e^{2z}}$.

(c) Find out the poles and the associated residues of $f(z)=z^3\sec z$.

II. EVALUATING THE INTEGRAL OF A COMPLEX FUNCTION

Example 4.3: Evaluate $\oint_C \frac{z}{4z^2+1}dz$ where C is a square of side 1 centered at $z=(1+i)/4$.

Solution:

The function $\frac{z}{4z^2+1}=\frac{z}{(2z+i)(2z-i)}$ is analytic except at $z_1=\frac{i}{2}$ and $z_2=-\frac{i}{2}$. However only z_1 is inside the circle. We have $Res(z=z_1)=\frac{i/2}{2(i/2)+i}=\frac{1}{2}$. Therefore $\oint_C \frac{z}{4z^2+1}dz=2\pi i Res(z_1)=\pi i$.

Home Work 4.2 Evaluate $\oint_C \frac{z^3e^{\pi z}}{(2z-i)^2}dz$ here C is $|z|=3$.

III. EVALUATING $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

Example 4.4: Evaluate $\int_0^{\pi} \frac{1}{3+\sin^2\theta}d\theta$

Solution:

First we have

$$\begin{aligned} \int_0^{\pi} \frac{1}{3+\sin^2\theta}d\theta = \int_{-\pi}^{0} \frac{1}{3+\sin^2\theta}d\theta &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{3+\sin^2\theta}d\theta. \\ = \frac{1}{2}\oint_{|z|=1} \frac{1}{3+\left(\frac{1}{2i}(z-\frac{1}{z})\right)^2}\frac{dz}{iz} \\ = 2i\oint_{|z|=1} \frac{z}{z^4-14z^2+1}\frac{dz}{iz} \end{aligned}$$

There are four roots of $z^4 - 14z^2 + 1$. They are $z_1=\sqrt{7+4\sqrt{3}}$, $z_2=\sqrt{7-4\sqrt{3}}$,$z_3=-\sqrt{7+4\sqrt{3}}$, $z_4=-\sqrt{7-4\sqrt{3}}$, only z_3 and z_4 are in the circle.

$$\begin{aligned} Resf(z_3) &= \frac{z_3}{(z_3-z_4)(z_3-z_2)(z_3-z_1)}, \\ Resf(z_2) &= \frac{z_2}{(z_2-z_3)(z_2-z_3)(z_2-z_1)}, \end{aligned}$$

Remember $z_1 = -z_3,\ z_2 = -z_4$ We have $Resf(z_3) + Resf(z_4) = \frac{1}{z_4^2-z_1^2} = \frac{\sqrt{3}}{24}$. Therefore

$$\int_0^{\pi} \frac{1}{3+\sin^2\theta}d\theta = (2\pi i)(2i)\left(\frac{\sqrt{3}}{24}\right) = \frac{\sqrt{3}\pi}{6}.$$

Home Work 4.3 Evaluate the following integrals:
(a) $\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}d\theta$. (b)$\int_0^{\pi} \sin^n\theta d\theta$.(c)$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta$ with $0 < b < a$.

IV. EVALUATING ALONG THE REAL AXIS

Example 4.5: Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx$

Solution:

We choose C is the path from $[-R, R]$ then goes to the semicircle enclosing the upper half plane. The singularity inside C is $z = ia$.

$$I = \oint_C \frac{1}{(z^2+a^2)^2}dz = \int_{-R}^{R} \frac{1}{(x^2+a^2)^2}dx + \int_{semicircle} \frac{1}{(z^2+a^2)^2}dz = (2\pi i)Resf(ia).$$

Γ: $z{=}ia+\rho e^{i\theta}$.

$$\begin{aligned}2\pi i Resf(ia) &= \oint_\Gamma \frac{1}{(z^2+a^2)^2}dz = \int_0^{2\pi}\frac{1}{(2ia+\rho e^{i\theta})^2\rho^2 e^{2i\theta}}\rho i e^{i\theta}d\theta \\
&= \frac{i}{(2ai)^2}\int_0^{2\pi}\left[1+\frac{\rho e^{i\theta}}{2ia}\right]^{-2}\frac{1}{\rho e^{i\theta}}d\theta \\
&= \frac{i}{(2ai)^2}\int_0^{2\pi}\frac{1}{\rho e^{i\theta}}\left[1-2\frac{\rho e^{i\theta}}{2ia}+\mathcal{O}(\rho^2)\right]d\theta \\
&= \frac{i}{-4a^2}\int_0^{2\pi}\left[\frac{1}{\rho}e^{-\theta}-\frac{2}{2ia}+\mathcal{O}(\rho)\right]d\theta \\
&= \frac{i}{-4a^2}(-\frac{4\pi}{2ia}) = \frac{\pi}{2a^3}.\end{aligned}$$

At the semicircle $z=Re^{i\theta}$.

$$\frac{1}{|z^2+a^2|^2} = \frac{1}{|R^2e^{2i\theta}+a^2|^2} \le \frac{1}{(R^2-a^2)^2}.$$

Since $|\int f(z)dz| \le \int |f(z)||dz|$. Hence

$$|\int\frac{dz}{(z^2+a^2)^2}| \le \int\frac{|dz|}{|z^2+a^2|^2} = \int_0^\pi\frac{Rd\theta}{|z^2+a^2|^2} \le \int_0^\pi\frac{Rd\theta}{(R^2-a^2)^2} = \frac{R\pi}{(R^2-a^2)^2}\to 0$$

as $R\to\infty$.

$$\begin{aligned}I &= \int_{-R}^{R}\frac{1}{(x^2+a^2)^2}dx+\int_{semicircle}\frac{1}{(z^2+a^2)^2}dz = \frac{\pi}{2a^3}. \\
&\Longrightarrow \lim_{R\to\infty}\left[\int_{-R}^{R}\frac{1}{(x^2+a^2)^2}dx+\int_{semicircle}\frac{1}{(z^2+a^2)^2}dz\right] \\
&= \int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}.\end{aligned}$$

Example 4.6: Evaluate $\int_{-\infty}^{\infty}\frac{\cos kx}{x^2+a^2}dx$

Solution:

$$\int_{-\infty}^{\infty}\frac{\cos kx}{x^2+a^2}dx = Re\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+a^2}dx\right].$$

Choose C_+ as $[-R,+R]$ combined with $R^{i\theta}$ with $0\le\theta\le\pi$. Due to Cauchy theorem we have

$$\oint_{C_+}\frac{e^{ikz}}{z^2+a^2}dz = \int_{-R}^{R}\frac{e^{ikx}}{x^2+a^2}dx+\int_{semicircle}\frac{e^{ikz}}{z^2+a^2}dz = 2\pi i Resf(ia).$$

Now we evaluate the integral along the semicircle:

$$\begin{aligned}|\int_{semicircle}\frac{e^{ikz}}{z^2+a^2}dz| &= |\int_0^\pi e^{ikR\cos\theta}e^{-kR\sin\theta}\frac{Rie^{i\theta}d\theta}{R^2e^{-2i\theta}+a^2}| \le \int_0^\pi |e^{ikR\cos\theta}e^{-kR\sin\theta}|\frac{|Rie^{i\theta}d\theta|}{|R^2e^{-2i\theta}+a^2|}\\ &= \int_0^\pi |e^{-kR\sin\theta}|\frac{|Rd\theta|}{|R^2-a^2|} < \frac{\pi R}{R^2-a^2}.\end{aligned}$$

On the other hand

$$Resf(ia) = \lim_{z\to ia}(z-ia)\frac{e^{ikz}}{z^2+a^2} = \frac{e^{-a}}{2ia}.$$

Therefore we have

$$\int_{-R}^{R}\frac{e^{ikx}}{x^2+a^2}dx + \int_{semicircle}\frac{e^{ikz}}{z^2+a^2}dz = \frac{e^{-a}\pi}{2a}.$$

$$\Longrightarrow \int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+a^2}dx = \frac{e^{-a}\pi}{2a}.$$

Taking the real part we have

$$\int_{-\infty}^{\infty}\frac{\cos kx}{x^2+a^2}dx = \frac{e^{-a}\pi}{2a}.$$

Home Work 4.4 Evaluate the following integrals:
(a) $\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+2x+2}dx$. (b)$\int_0^\infty \frac{x^6}{(x^4+1)^2}dx$. (c)$\int_0^\infty \frac{1}{(x^4+1)^2}dx$.

V. JORDAN LEMMA

If $f(z)$ converges uniformly to zero whenever $z\to\infty$ then

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}dz = 0.$$

here C_R is the upper half of the circle $|z| = R$.

Proof:

$f(z)$ converges uniformly to zero whenever $z\to\infty$ means that when R is large than certain M $|f(z)|\le\epsilon$.

$$\begin{aligned}&|\int_{C_R} f(z)e^{ikz}dz| = |\int_0^\pi f(z)e^{ikR\cos\theta}e^{-kR\sin\theta}Rie^{i\theta}d\theta|\\ &\le \int_0^\pi |f(z)|e^{-kR\sin\theta}Rd\theta \le \int_0^\pi \epsilon e^{-kR\sin\theta}Rd\theta \le \epsilon R\int_0^\pi \epsilon e^{-kR\sin\theta}d\theta\end{aligned}$$

However if we set $\phi=\pi-\theta$ then $\int_{\pi/2}^{\pi} e^{-kR\sin\theta}d\theta=\int_{\pi/2}^{0} e^{-kR\sin(\pi-\phi)}(-d\phi)=\int_{0}^{\pi/2} e^{-kR\sin\phi}d\phi$. Thus

$$|\int_{C_R} f(z)e^{ikz}dz| \leq 2\epsilon R\int_0^{\pi/2} \epsilon e^{-kR\sin\theta}d\theta$$

Furthermore we have

$$\sin\theta \geq \frac{2\theta}{\pi} \Longrightarrow 0 \leq \theta \leq \frac{\pi}{2}.$$

This can be seen by plotting the two curves to compare. So at end we have

$$|\int_{C_R} f(z)e^{ikz}dz| \leq \frac{\epsilon R\pi}{kR}\left(1-e^{-kR}\right) \leq \frac{\epsilon\pi}{k} \longrightarrow 0.\ as\ \epsilon \longrightarrow 0.$$

Since ϵ can be arbitrary small so the lemma is proved.

VI. RECTANGLE AS THE CONTOUR

Example 4.7: Evaluate $\int_{-\infty}^{\infty}\frac{1}{\cosh kx}dx$

Solution:

Choose C as combination of $L_1=[-R,+R]$, $L_2=[R,R+i\frac{\pi}{k}]$,$L_3=[R+i\frac{\pi}{k},-R+i\frac{\pi}{k}]$ and $L_4=[-R+i\frac{\pi}{k},-R]$. From Cauchy theorem

$$\oint_C \frac{1}{\cosh kz}dz = 2\pi i Res\left(\frac{i\pi}{2k}\right).$$

Here

$$Resf\left(\frac{i\pi}{2k}\right) = \lim_{z\to\frac{i\pi}{2k}}\frac{(z-i\frac{i\pi}{2k})}{\cosh kz} = \frac{1}{k\sinh\left(\frac{i\pi}{2k}\right)} = \frac{1}{ik}.$$

Since $\cosh k(z+i\frac{\pi}{k}) = -\cosh kz$ hence $\int_{L_1}\frac{1}{\cosh kz}dz=\int_{L_3}\frac{1}{\cosh kz}dz=\int_{-R}^{R}\frac{1}{\cosh kx}dx$. On the other hand, at L_2 one has $z=R+iy$, $\frac{1}{\cosh kz} = \frac{2}{e^{iky}e^{kR}+e^{-iky}e^{-kR}}$ thus

$$|\int_0^{\frac{\pi}{k}}\frac{2idy}{e^{iky}e^{kR}+e^{-iky}e^{-kR}}| \leq \int_0^{\frac{\pi}{k}}\frac{|2idy|}{|e^{iky}e^{kR}+e^{-iky}e^{-kR}|} \leq \int_0^{\frac{\pi}{k}}\frac{|2dy|}{|e^{kR}-e^{-kR}|} < \int_0^{\frac{\pi}{k}}\frac{|2dy|}{|e^{kR}|} = \frac{2\pi}{ke^{kR}}.$$

Hence $|\int_{L_2}\frac{1}{\cosh kz}dz| \leq \frac{2\pi}{ke^{kR}}$.

Similarly at L_2 one has $z=-R+iy$ hence $\frac{1}{\cosh kz} = \frac{2}{e^{iky}e^{-kR}+e^{-iky}e^{kR}}$ thus

$$|\int_0^{\frac{\pi}{k}}\frac{2idy}{e^{iky}e^{-kR}+e^{-iky}e^{kR}}| \leq \int_0^{\frac{\pi}{k}}\frac{|2idy|}{|e^{iky}e^{-kR}+e^{-iky}e^{kR}|} \leq \int_0^{\frac{\pi}{k}}\frac{|2dy|}{|e^{-kR}-e^{kR}|} < \int_0^{\frac{\pi}{k}}\frac{|2dy|}{|e^{kR}|} = \frac{2\pi}{ke^{kR}}.$$

Hence $|\int_{L_4}\frac{1}{\cosh kz}dz| \leq \frac{2\pi}{ke^{kR}}$.

As $R \to \infty$ we have $|\int_{L_2,L_4} \frac{1}{\cosh kz} dz| \to 0$. So that we have

$$\oint_C \frac{1}{\cosh kz} dz = \left(\int_{L_1} dz + \int_{L_3} dz\right) \frac{1}{\cosh kz} dz = 2\int_{-\infty}^{\infty} \frac{1}{\cosh kx} dx = 2\pi i \frac{1}{ik} = \frac{2\pi}{k}.$$

Hence $\int_{-\infty}^{\infty} \frac{1}{\cosh kx} dx = \frac{\pi}{k}$.

Home Work 4.5:Evaluate $\int_0^\infty \frac{(\ln x)^2}{x^2+1} dx$

Home Work 4.6:Evaluate $\int_0^\infty \frac{cosx}{e^x+e^{-x}} dx$

Home Work 4.7:Evaluate $\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx$, $0 < a < 1$.

VII. INTEGRAL OF MULTI-VALUED FUNCTIONS

Example 4.8: Evaluate $\int_0^\infty \frac{\sqrt{x}}{x^2+1} dx$

Choose C as combination of $L_1 : z = r + i\rho \sin\epsilon, \rho\cos\epsilon \le r \le R$, $L_2 : z = Re^{i\theta}, \epsilon' \le \theta \le 2\pi - \epsilon'$, $L_3 : z = re^{2\pi i} - i\rho\sin\epsilon, \rho\cos\epsilon \le r \le R$ and $L_4 = \rho e^{i\theta}, \epsilon \le \theta \le 2\pi - \epsilon$.

$$\begin{aligned} Resf(+i) &= \frac{i^{1/2}}{2i} = \frac{\sqrt{2}}{4i}(1+i). \\ Resf(-i) &= \frac{(-i)^{1/2}}{-2i} = \frac{\sqrt{2}}{4i}(1-i) \end{aligned}$$

$$\oint_C \frac{\sqrt{z}}{z^2+1} dz = \pi\sqrt{2}.$$

$$\int_{L_1} \frac{\sqrt{z}}{z^2+1} dz = \int_{\rho\cos\epsilon}^{R} \frac{\sqrt{r + i\rho\sin\epsilon}}{(r+i\rho\sin\epsilon)^2+1} dr.$$

$$\int_{L_3} \frac{\sqrt{z}}{z^2+1} dz = \int_{R}^{\rho\cos\epsilon} \frac{\sqrt{re^{2\pi i} - i\rho\sin\epsilon}}{(r-i\rho\epsilon)^2+1} dr$$

$$\int_{L_4} \frac{\sqrt{z}}{z^2+1} dz = \int_{2\pi-\epsilon}^{\epsilon} \frac{\sqrt{\rho e^{i\theta}}}{\rho^2 e^{2i\theta}+1} \rho i e^{i\theta} d\theta = \int_{2\pi-\epsilon}^{\epsilon} \frac{i\rho^{3/2} e^{i3\theta/2}}{\rho^2 e^{2i\theta}+1} d\theta \to 0.$$

As $\rho \to 0$. Furthermore we have $\int_{L_1} \frac{\sqrt{z}}{z^2+1} dz = \int_{L_3} \frac{\sqrt{z}}{z^2+1} dz = \int_0^R \frac{\sqrt{x}}{x^2+1} dx$. Besides

$$|\int_{L_2} \frac{\sqrt{z}}{z^2+1} dz| = |\int_{\epsilon'}^{2\pi-\epsilon'} \frac{\sqrt{R} e^{i\theta/2}}{R^2 e^{2i\theta}+1} Rie^{i\theta} d\theta| \le \int_{\epsilon}^{2\pi-\epsilon'} \frac{|\sqrt{R}|}{|R^2-1|} |Rd\theta| = \frac{R^{3/2}}{R^2-1}(2\pi - 2\epsilon') \to 0.$$

as $R \to \infty$. Therefore

$$\lim_{R\to\infty} \oint_C \frac{\sqrt{z}}{z^2+1} dz = 2\int_0^\infty \frac{\sqrt{x}}{x^2+1} dx = \sqrt{2}\pi.$$

Hence $\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi}{\sqrt{2}}$.

Example 4.9 Evaluate $\int_0^\infty \frac{x^{a-1}}{x+1}dx$. Here $0 \le a \le 1$.

Solution:

Choose C=$C_1 + C_2 + C_3$+C_4. Here C_1=$[\rho\cos\varepsilon + i\rho\sin\varepsilon, R + i\rho\varepsilon]$, C_2: $z = Re^{i\theta}, \epsilon' \le \theta \le 2\pi - \epsilon'$. C_3:$[R - i\rho\sin\varepsilon, \rho\cos\varepsilon - i\rho\sin\varepsilon]$. The only singularity is at $z = -1$:

$$\oint_C \frac{z^{a-1}}{z+1}dz = 2\pi i Res(z=-1) = 2\pi i e^{i(a-1)\pi} = -2\pi i e^{ia\pi}.$$

The integral of C_2 approaches zero when $R \to \infty$ since

$$\begin{aligned} |\int_{C_2} \frac{z^{a-1}}{z+1}dz| &= |\int_0^{2\pi} \frac{R^{a-1}e^{i(a-1)\theta}}{Re^{i\theta}+1}Rie^{i\theta}d\theta| \\ &\le \int_0^{2\pi} \frac{|R^{a-1}e^{i(a-1)\theta}|}{|Re^{i\theta}+1|}|Rie^{i\theta}|d\theta = \int_0^{2\pi} \frac{R^a}{R-1}d\theta = 2\pi\frac{R^a}{R-1} \to 0. \end{aligned}$$

as $R \to \infty$.

The integral of C_4 also approaches zero when $\rho \to 0$ since

$$\begin{aligned} |\int_{C_4} \frac{z^{a-1}}{z+1}dz| &= |\int_\varepsilon^{2\pi-\varepsilon} \frac{\rho^{a-1}e^{i(a-1)\theta}}{\rho e^{i\theta}+1}\rho i e^{i\theta}d\theta| \\ &\le \int_0^{2\pi} \frac{|\rho^{a-1}e^{i(a-1)\theta}|}{|\rho e^{i\theta}+1|}|\rho i e^{i\theta}|d\theta = \int_0^{2\pi} \frac{\rho^a}{1-\rho}d\theta \le 2\pi\rho^a \to 0. \end{aligned}$$

as $\rho \to 0$. Under the limit $\rho \to 0$ and along the branch cut we have

$$\int_{C_3} \frac{z^{a-1}}{z+1}dz = \int_R^0 \frac{\rho^{a-1}e^{i(a-1)2\pi}}{1+\rho}d\rho = -e^{i(a-1)2\pi}\int_0^R \frac{\rho^{a-1}}{1+\rho}d\rho = -e^{i(a-1)2\pi}\int_{C_1} \frac{z^{a-1}}{z+1}dz.$$

When we take limits: $R \to \infty, \rho \to 0$, we obtain

$$(1 - e^{i(a-1)2\pi})\int_0^\infty \frac{\rho^{a-1}}{1+\rho}d\rho = -2\pi i e^{ia\pi}.$$

$$\int_0^\infty \frac{\rho^{a-1}}{1+\rho}d\rho = \frac{-2\pi i e^{ia\pi}}{1-e^{i(a-1)2\pi}} = \frac{-2\pi i}{e^{-ia\pi}-e^{ia\pi}} = \frac{-2\pi i}{-2i\sin a\pi} = \frac{\pi}{\sin a\pi}.$$

Example 4.10 Evaluate $\int_0^\infty \frac{x^{a-1}\ln x}{x+1}dx$.

Solution:

C=C_1+C_2+C_3+C_4. C_1=$[\rho e^\epsilon, R + i\rho\sin\epsilon = Re^{i\epsilon'}]$. C_2=$Re^{i\theta}, i\epsilon' \le \theta \le 2\pi - i\epsilon'$.

C_3=$[Re^{2\pi-i\epsilon} = R - i\rho\sin\epsilon, \rho e^{-i\epsilon}]$. C_4=$\rho e^{i\theta}, \epsilon \le \theta \le 2\pi - \epsilon$. The only singularity is at $z = -1$. So that

$$\oint_C \frac{z^{a-1}\ln z}{z+1}dz = 2\pi i Res(z=-1) = 2\pi i e^{i(a-1)\pi}\ln(-1) = 2\pi i(-e^{ia\pi})(i\pi) = 2\pi^2 e^{ia\pi}.$$

$$\begin{aligned}
|\int_{C_2}\frac{z^{a-1}\ln z}{z+1}dz| &= |\int_0^{2\pi}\frac{R^{a-1}e^{i(a-1)\theta}(\ln R + i\theta)}{Re^{i\theta}+1}Rie^{i\theta}d\theta| \\
&\le \int_0^{2\pi}\frac{|R^{a-1}e^{i(a-1)\theta}|\sqrt{(\ln R)^2+4\pi^2}}{|Re^{i\theta}+1|}|Rie^{i\theta}|d\theta \\
&= \int_0^{2\pi}\frac{R^a\sqrt{(\ln R)^2+4\pi^2}}{R-1}d\theta = 2\pi\frac{R^a}{R-1}\sqrt{(\ln R)^2+4\pi^2} \\
&\le 4\sqrt{2}\pi^2\frac{R^a\ln R}{R-1} = \frac{4\sqrt{2}\pi^2 x}{1-x}\frac{\ln x}{x^a} \to 0.
\end{aligned}$$

as $x = \frac{1}{R} \to 0$. Here we choose $\ln R > 2\pi$. On the other hand,

$$\begin{aligned}
|\int_{C_4}\frac{z^{a-1}\ln z}{z+1}dz| &= |\int_0^{2\pi}\frac{\rho^{a-1}e^{i(a-1)\theta}(\ln\rho + i\theta)}{\rho e^{i\theta}+1}\rho ie^{i\theta}d\theta| \\
&\le \int_0^{2\pi}\frac{|\rho^{a-1}e^{i(a-1)\theta}|\sqrt{(\ln\rho)^2+\pi^2}}{|\rho e^{i\theta}+1|}|\rho ie^{i\theta}|d\theta \\
&= \int_0^{2\pi}\frac{\rho^a\sqrt{(\ln\rho)^2+4\pi^2}}{1-\rho}d\theta \le 2\pi\rho^a\sqrt{(\ln\rho)^2+4\pi^2} \\
&\le 4\sqrt{2}\pi^2\rho^a\ln\rho \to 0.
\end{aligned}$$

as $\rho \to 0$. Here we choose $\ln\rho > 4\pi^2$. Since $\int_{C_2} \to 0$ as $R \to \infty$ and $\int_{C_4} \to 0$ as $\rho \to 0$.

$$\begin{aligned}
&(1-e^{i(a-1)2\pi})\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho - 2\pi ie^{i(a-1)2\pi}\int_0^\infty\frac{\rho^{a-1}}{\rho+1}d\rho = 2\pi^2e^{ia\pi}. \\
\Longrightarrow\ &(1-e^{i(a-1)2\pi})\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho = (2\pi i)e^{i2a\pi}\frac{\pi}{\sin a\pi} + 2\pi^2e^{ia\pi}. \\
\Longrightarrow\ &(1-e^{i2a\pi})\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho = \frac{2i\pi^2e^{i2a\pi}}{\sin a\pi} + 2\pi^2e^{ia\pi}. \\
\Longrightarrow\ &(e^{-ia\pi}-e^{ia\pi})\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho = \frac{2i\pi^2e^{ia\pi}}{\sin a\pi} + 2\pi^2. \\
\Longrightarrow\ &-2i\sin a\pi\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho = \frac{2i\pi^2(e^{ia\pi}-i\sin a\pi)}{\sin a\pi}. \\
\Longrightarrow\ &\int_0^\infty\frac{\rho^{a-1}\ln\rho}{\rho+1}d\rho = \frac{-\pi^2(e^{-ia\pi}-i\sin a\pi)}{\sin^2 a\pi} = \frac{-\pi^2\cos a\pi}{\sin^2 a\pi}
\end{aligned}$$

The following problems are left for exercise.

Home Work 4.8 Evaluate $\int_0^\infty \frac{(\ln x)}{(x^2+1)^2}dx$.

Home Work 4.9 Evaluate $\int_0^1 x^a(1-x)^{1-a}dx$. Here $-1 \le a \le 2$.

Home Work 4.10: Evaluate $\int_0^\infty \frac{x^a}{x^2+1}dx$. $0 \le a \le 1$.

Home Work 4.11: Evaluate $\int_0^\infty \frac{\ln(1+x^2)}{x^{1+a}}dx$. $0 \le a \le 2$.

VIII. SPECIAL EXAMPLES

Example 4.11: Evaluate $\int_0^\infty \sin x^2 dx$

Solution:

Choose C=C_1+C_2+C_3. Here C_1=$[0,R]$, C_2:$z = Re^{i\theta}, 0 \le \theta \le \frac{\pi}{4}$. C_3:$z = \rho e^{i\pi/4},\ 0 \le \rho \le R$. Since e^{iz^2} is analytic in and on C so that

$$\oint_C e^{iz^2}dz = \int_{C_1} e^{iz^2}dz + \int_{C_2} e^{iz^2}dz + \int_{C_3} e^{iz^2}dz = 0.$$

$$\begin{aligned} |\int_{C_2} e^{iz^2}dz| &= |\int_0^{\pi/4} e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta} iRe^{i\theta}d\theta| \le \int_0^{\pi/4} |e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}||iRe^{i\theta}|d\theta \\ &= \int_0^{\pi/4} |e^{-R^2\sin 2\theta}||R|d\theta = \frac{R}{2}\int_0^{\pi/2} |e^{-R^2\sin\theta}|d\theta \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2 2\theta/\pi}d\theta \\ &= \frac{\pi}{4R}\left(1-e^{-R^2}\right) \to 0 \end{aligned}$$

as $R \to \infty$. On the other hand

$$\int_{C_3} e^{iz^2}dz = \int_R^0 \exp(-\rho^2)e^{i\pi/4}d\rho = -e^{i\pi/4}\int_0^R \exp(-\rho^2)d\rho.$$

Hence

$$\int_0^\infty e^{i\rho^2}d\rho = e^{i\pi/4}\int_0^\infty \exp(-\rho^2)d\rho.$$

Since $\int_0^\infty \exp(-\rho^2)d\rho$=$\frac{\sqrt{\pi}}{2}$. Hence $\int_0^\infty e^{i\rho^2}d\rho$=$\frac{1+i}{\sqrt{2}}\frac{\sqrt{\pi}}{2}$. Therefore $\int_0^\infty \sin x^2 dx$=$\frac{\pi}{2\sqrt{2}}$.

Example 4.12: Evaluate $\int_0^\infty \frac{\sin kx}{x}dx$

Solution:

$\int_0^\infty \frac{\sin kx}{x}dx = \int_{-\infty}^0 \frac{\sin kx}{x}dx$. Now we have $\int_0^\infty \frac{\sin kx}{x}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin kx}{x}dx = Im\frac{1}{2}\int_{-\infty}^\infty \frac{e^iz}{z}dz$. Choose C as combination of C_1=$[-R,-\rho]$ and $C_2 : z = \rho e^{i\theta}, -\pi \le \theta \le \pi$ and C_3=$[\rho,R]$ and $C_4 : z = Re^{i\theta}, 0 \le \theta \le \pi$. From Cauchy theorem we have

$$\oint_C \frac{e^{ikz}}{z}dz = 0.$$

First we evaluate the integral of C_4:

$$|\int_{C_4} \frac{e^{ikz}}{z}dz| = |\int_0^{\pi} \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{Re^{i\theta}}Rie^{i\theta}d\theta| \leq \int_0^{\pi} \frac{|e^{ikR\cos\theta}e^{-kR\sin\theta}|}{|Re^{i\theta}|}|Rie^{i\theta}d\theta|$$
$$= \int_0^{\pi} e^{-kR\sin\theta}d\theta \leq \frac{\epsilon\pi}{R} = 2\int_0^{\pi/2} e^{-kR\sin\theta}d\theta \leq 2\int_0^{\pi/2} e^{-2kR\theta/\pi}d\theta = \frac{\pi}{kR}\left(1-e^{-kR}\right).$$

As $R \to \infty$, $\int_{C_4} \to 0$.

$$\int_{C_2} \frac{e^{ikz}}{z}dz = \int_{\pi}^{0} \frac{e^{i\rho\cos\theta}e^{-\rho\sin\theta}}{\rho e^{i\theta}}\rho ie^{i\theta}d\theta = i\int_{\pi}^{0} e^{i\rho\cos\theta}e^{-\rho\sin\theta}d\theta.$$

So we have

$$\int_{-\infty}^{-\rho} \frac{e^{ikx}}{x}dx + i\int_{\pi}^{0} e^{i\rho\cos\theta}e^{-\rho\sin\theta}d\theta + \int_{\rho}^{\infty} \frac{e^{ikx}}{x}dx \to 0.$$

Next, we take the limit: $\rho \to 0$ then

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}dx = \pi i.$$

Therefore $\int_{\infty}^{\infty} \frac{e^{ikx}}{x}dx = i\frac{\pi}{2}$. Thus taking the imaginary part one obtains $\int_{-\infty}^{\infty} \frac{\sin kx}{x}dx = \pi$. Therefore we have $\int_0^{\infty} \frac{\sin kx}{x}dx = \frac{\pi}{2}$ since it is even function.

Example: 4.13 Evaluate the value of $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}dx$.

$\frac{\exp(iz)}{z^3}$ is an analytic function except at $z = 0$.

$$\oint \frac{\exp(iz)}{z^3}dz = \int_{-R}^{-\rho} \frac{\exp(ix)}{x^3}dx + \int_{C_\rho} \frac{\exp(iz)}{z^3}dz + \int_{\rho}^{R} \frac{\exp(ix)}{x^3}dx + \int_{C_R} \frac{\exp(iz)}{z^3}dz = 0.$$

Because

$$\int_{-R}^{-\rho} \frac{\exp(ix)}{x^3}dx = \int_{R}^{\rho} \frac{\exp(-ix)}{x^3}dx = -\int_{\rho}^{R} \frac{\exp(-ix)}{x^3}dx.$$

Therefore

$$\oint \frac{\exp(iz)}{z^3}dz = \int_{C_\rho} \frac{\exp(iz)}{z^3}dz + \int_{\rho}^{R} \frac{\exp(ix)-\exp(-ix)}{x^3}dx + \int_{C_R} \frac{\exp(iz)}{z^3}dz = 0.$$

When $R \to \infty$ then we know $\int_{C_R}$ is zero. We have

$$\int_{C_\rho} \frac{\exp(iz)}{z^3}dz + \int_{\rho}^{\infty} \frac{\exp(ix)-\exp(-ix)}{x^3}dx = 0.$$

Now we have to evaluate

$$\int_{C_\rho} \frac{\exp(iz)}{z^3}dz = \int_{\pi}^{0} \frac{\exp(i\rho\exp(i\theta))}{\rho^3\exp(i3\theta)}i\rho\exp(i\theta)d\theta = -i\int_0^{\pi} \frac{\exp(i\rho\exp(i\theta))}{\rho^2\exp(i2\theta)}d\theta$$

$$= -i\int_0^\pi \frac{1}{\rho^2\exp(i2\theta)}d\theta + \int_0^\pi \frac{\rho\exp(i\theta)}{\rho^2\exp(i2\theta)}d\theta + \frac{i}{2}\int_0^\pi \frac{\rho^2\exp(2i\theta)}{\rho^2\exp(i2\theta)}d\theta + \mathcal{O}(\rho)$$
$$= -i\int_0^\pi \rho^{-2}\exp(-i2\theta)d\theta + \int_0^\pi \rho^{-1}\exp(-i\theta)d\theta + \frac{i}{2}\int_0^\pi d\theta + \mathcal{O}(\rho)$$
$$= -i\frac{\exp(-2i\pi)-\exp(0)}{-2i\rho^2} + \frac{\exp(-i\pi)-\exp(0)}{-i\rho} + \frac{i\pi}{2} + \mathcal{O}(\rho)$$
$$= -i\frac{2}{\rho} + \frac{i\pi}{2} + \mathcal{O}(\rho).$$

So
$$2i\int_\rho^\infty \frac{\sin x}{x^3}dx = i\frac{2}{\rho} - \frac{i\pi}{2} + \mathcal{O}(\rho).$$

That is
$$\int_\rho^\infty \frac{\sin x}{x^3}dx = \frac{1}{\rho} - \frac{\pi}{4} + \mathcal{O}(\rho).$$

Similarly we have
$$\int_{C_\rho} \frac{\exp(i3z)}{z^3}dz + \int_\epsilon^\infty \frac{\exp(i3x)-\exp(-i3x)}{x^3}dx = 0.$$

$$\int_{C_\rho} \frac{\exp(i3z)}{z^3}dz = \int_\pi^0 \frac{\exp(i3\rho\exp(i\theta))}{\rho^3\exp(i3\theta)}i\rho\exp(i\theta)d\theta = -i\int_0^\pi \frac{\exp(i3\rho\exp(i\theta))}{\rho^2\exp(i2\theta)}d\theta$$
$$= -i\int_0^\pi \frac{1}{\rho^2\exp(i2\theta)}d\theta + 3\int_0^\pi \frac{\rho\exp(i\theta)}{\rho^2\exp(i2\theta)}d\theta + \frac{9i}{2}\int_0^\pi \frac{\rho^2\exp(2i\theta)}{\rho^2\exp(i2\theta)}d\theta + \mathcal{O}(\rho)$$
$$= -i\int_0^\pi \rho^{-2}\exp(-i2\theta)d\theta + 3\int_0^\pi \rho^{-1}\exp(-i\theta)d\theta + \frac{9i}{2}\int_0^\pi d\theta + \mathcal{O}(\rho)$$
$$= -i\frac{\exp(-2i\pi)-\exp(0)}{-2i\rho^2} + 3\frac{\exp(-i\pi)-\exp(0)}{-i\rho} + \frac{i9\pi}{2} + \mathcal{O}(\rho)$$
$$= -i\frac{6}{\rho} + \frac{i9\pi}{2} + \mathcal{O}(\rho).$$

So
$$2i\int_\rho^\infty \frac{\sin 3x}{x^3}dx = i\frac{6}{\rho} - \frac{9i\pi}{2} + \mathcal{O}(\rho).$$

That is
$$\int_\rho^\infty \frac{\sin 3x}{x^3}dx = \frac{3}{\rho} - \frac{9\pi}{4} + \mathcal{O}(\rho).$$

Now
$$\int_\rho^\infty \frac{\sin^3 x}{x^3}dx = \frac{3}{4}\int_\rho^\infty \frac{\sin x}{x^3}dx - \frac{1}{4}\int_\rho^\infty \frac{\sin 3x}{x^3}dx = \frac{3}{4}\left(\frac{1}{\rho} - \frac{\pi}{4}\right) - \frac{1}{4}\left(\frac{3}{\rho} - \frac{9\pi}{4}\right) = \frac{3\pi}{8} + \mathcal{O}(\rho)$$

Take $\rho \to 0$ we have
$$\int_0^\infty \frac{\sin^3 x}{x^3}dx = \frac{3\pi}{8}.$$

Example: 4.14 Evaluate the following integral: $\int_0^\infty \frac{1}{1+x^3}dx$.

Solution:

Choose C=C_1+C_2+C_3. C_1=$[0, R]$, C_2=$Re^{i\theta}$, $\frac{4\pi}{3} \leq \theta \leq 2\pi$. C_3=$re^{i4/3\pi}, 0 \leq r \leq R$. WE first have

$$\oint_C \frac{dz}{1+z^3} = -2\pi i Res(z = e^{i5/3\pi}) = -2\pi i \frac{1}{(e^{i5/3\pi}-1)(e^{i5/3\pi}-e^{i/3\pi})} = -2\pi i \frac{1}{3e^{i2\pi/3}}.$$

The integral of C_2 is bounded by

$$|\int_{C_2} \frac{dz}{1+z^3}| = |\int_{2\pi}^{4\pi/3} d\theta \frac{Rie^{i\theta}d\theta}{1+R^3e^{i3\theta}}| = |\int_{4\pi/3}^{2\pi} d\theta \frac{Rie^{i\theta}}{1+R^3e^{i3\theta}}|$$
$$\leq \int_{4\pi/3}^{2\pi} d\theta \frac{|Rie^{i\theta}|}{|1+R^3e^{i3\theta}|} \leq \int_{4\pi/3}^{2\pi} d\theta \frac{R}{R^3-1} \to 0.$$

when $R \to \infty$. Furthermore we have

$$\int_{c_3} \frac{dz}{1+z^3} = \int_R^0 dr \frac{e^{i4\pi/3}dr}{1+r^3} = e^{i4\pi/3} \int_R^0 dr \frac{1}{1+r^3} dr = -e^{i4\pi/3} \int_0^R dr \frac{1}{1+r^3} dr.$$

Therefore when $R \to \infty$ we have $\int_{C_1} + \int_{C_3}$=$(1-e^{i4\pi/3}) \int_0^\infty \frac{dr}{1+r^3}$=$-2\pi i \frac{1}{3e^{i2\pi/3}}$. Hence we have

$$\int_0^\infty \frac{dr}{1+r^3} = \frac{-2\pi i \frac{1}{3e^{i2\pi/3}}}{1-e^{i4\pi/3}} = \frac{2\pi i}{3(e^{i2\pi/3}-e^{-i2\pi/3})} = \frac{2\pi i}{6i \sin \frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}}.$$

Home Work 4.12: Evaluate $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ for positive integrs m and n. $n > m$.

Example: 4.15 Evaluate the following integral: $\int_{0^+}^{\pi^-} \ln \sin x dx$.

Solution:

We start to evaluate the following integral: $I = \int_C \ln(1-e^{2iz})dx$. The contour C contains six sections: C_1 is 1/4 circle around the origin. C_2 is the real axis from ε to $\pi - \varepsilon$. C_3 is 1/4 circle around π. C_4 is parallel to imaginary axis from $\pi + i\varepsilon$ to $\pi + iY$. Y is an arbitrary positive number. C_5 is parallel to the real axis from $\pi + iY$ to iY. C_6 is imaginary axis from iY to $i\varepsilon$. Because no singularity is inside C therefore $I = 0$ according to Cauchy theorem:

$$I = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} + \int_{C_6} \ln(1-e^{2iz})dx \right) = 0$$

On C_1, $z = \varepsilon e^{i\theta}$. We have

$$\int_{C_1} \ln(1-e^{2iz})dz = \int_{\pi/2}^0 \ln(1-e^{-2\varepsilon \sin\theta} e^{2i\varepsilon \cos\theta}) \varepsilon i e^{i\theta} d\theta.$$

Because $\ln z = \ln|z| + iargz$, one has $|\ln z| = \sqrt{(\ln|z|)^2 + (argz)^2} \leq \sqrt{(\ln|z|)^2 + 4\pi^2}$.

Hence one has

$$|\ln(1 - e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})| \leq \sqrt{\left(\ln\left[\sqrt{1 + e^{-4\varepsilon\sin\theta} - 2e^{-2\varepsilon\sin\theta}\cos(2\varepsilon\cos\theta)}\right]\right)^2 + 4\pi^2}$$

$$\leq \sqrt{\left(\ln\left[\sqrt{1 + e^{-4\varepsilon\sin\theta} - 2e^{-2\varepsilon\sin\theta}}\right]\right)^2 + 4\pi^2} = \sqrt{(\ln|1 - e^{-2\varepsilon\sin\theta}|^2)^2 + 4\pi^2}$$

$$\leq 2|\ln|1 - e^{-2\varepsilon\sin\theta}|| + 2\pi$$

Therefore

$$|\int_{C_1}\ln(1-e^{2iz})dz| = |\int_{\pi/2}^{0}\ln(1-e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}d\theta| \leq \int_0^{\pi/2}|\ln(1-e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}|d\theta$$

$$= \int_0^{\pi/2}|\ln(1 - e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})|\varepsilon d\theta \leq \int_0^{\pi/2}2(|\ln|1 - e^{-2\varepsilon\sin\theta}|| + 2\pi)\varepsilon d\theta$$

$$= 2\varepsilon\int_0^{\pi/2}(\ln[(1 - e^{-2\varepsilon\sin\theta})^{-1}] + \pi)d\theta$$

However

$$\int_0^{\pi/2}\ln[(1 - e^{-2\varepsilon\sin\theta})^{-1}]\varepsilon d\theta \leq \int_0^{\pi/2}\ln[(1 - e^{-4\varepsilon\theta/\pi})^{-1}]\varepsilon d\theta$$

$$= \int_0^{\pi/2}\left[\sum_n \frac{e^{-4n\varepsilon\theta}}{n}\right]\varepsilon d\theta = \sum_{n=1}\frac{1}{n}\int_0^{\pi/2}e^{-4n\varepsilon\theta/\pi}\varepsilon d\theta$$

$$= \sum_{n=1}\frac{\pi}{4n^2}\left(1 - e^{-2n\varepsilon}\right) \leq \sum_{n=1}\frac{\pi}{4n^2}\sqrt{2n\varepsilon} = \varepsilon^{1/2}\sum_{n=1}\frac{\pi\sqrt{2}}{n^{3/2}} = \varepsilon^{1/2}\pi\sqrt{2}\zeta(3) \to 0.$$

as $\varepsilon \longrightarrow 0$. Here we use the identity $1 - e^{-x} - \sqrt{x}, 0$ for $x \geq 0$. $\zeta(s)$ is Riemann Zeta function and $\zeta(3)$=2.66...< 3. Hence $\lim_{\varepsilon\to 0}\int_{C_1}\ln(1 - e^{2iz})dz \longrightarrow 0$.

Similarly on C_3, $z = \pi + \varepsilon e^{i\theta}$,

$$\int_{C_3}\ln(1-e^{2iz})dz = \int_\pi^{\pi/2}\ln(1-e^{2i\pi}e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}d\theta = \int_\pi^{\pi/2}\ln(1-e^{-2\varepsilon\sin\theta}e^{2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}d\theta.$$

$$= \int_{\pi/2}^{0}\ln(1 - e^{-2\varepsilon\sin\theta}e^{-2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}d\theta.$$

Therefore we have

$$|\int_{C_3}\ln(1 - e^{2iz})dz| = |\int_{\pi/2}^{0}\ln(1 - e^{-2\varepsilon\sin\theta}e^{-2i\varepsilon\cos\theta})\varepsilon ie^{i\theta}d\theta| \leq$$

$$= 2\varepsilon\int_0^{\pi/2}(\ln[(1 - e^{-2\varepsilon\sin\theta})^{-1}] + \pi)d\theta \to 0$$

as $\varepsilon \longrightarrow 0$. Hence $\lim_{\varepsilon\to 0}\int_{C_3} \ln(1-e^{2iz})dz \longrightarrow 0$.

On C_4, $z=\pi+iy$ and

$$\int_{C_4} \ln(1-e^{2iz})dz = \int_{\varepsilon}^{Y} \ln(1-e^{2i\pi}e^{-2y})idy = i\int_{\varepsilon}^{Y} \ln(1-e^{-2y})dy.$$

On the other hand, on C_6. $z=iy$ and

$$\int_{C_6} \ln(1-e^{2iz})dz = \int_{Y}^{\varepsilon} \ln(1-e^{-2y})idy = -i\int_{\varepsilon}^{Y} \ln(1-e^{-2y})dy = -\int_{C_4} \ln(1-e^{2iz})dz.$$

On C_5, $z=x+iY$. One has

$$\int_{C_5} \ln(1-e^{2iz})dz = \int_{\pi}^{0} \ln(1-e^{2ix}e^{-2Y})dx = -\int_0^{\pi} \sum_{N=1}^{N=\infty} \frac{-1}{N} e^{-2NY}e^{2iNx}dx$$

$$= \sum_{N=1}^{N=\infty} e^{-2NY}\int_0^{\pi} e^{2iNx}dx = 0$$

Because $\int_0^{\pi} e^{2iNx}dx = 0$. Therefore $\int_{C_5} \ln(1-e^{2iz})dz = 0$

Since $\lim_{\varepsilon\to 0}\int_{C_1} = \lim_{\varepsilon\to 0}\int_{C_3} = 0$, $\int_{C_4} + \int_{C_6} = 0$ and $\int_{C_5} = 0$. Therefore we have

$$\lim_{\varepsilon\to 0}\int_{C_2} \ln(1-e^{2iz})dz = \int_{\varepsilon}^{\pi-\varepsilon} \ln(1-e^{2ix})dx = 0.$$

We evaluate this integral as follows:

$$\int_{\varepsilon}^{\pi-\varepsilon} \ln(1-e^{2ix})dx = \int_{\varepsilon}^{\pi-\varepsilon} \ln(e^{ix}(-2i)\sin x dx = \int_{\varepsilon}^{\pi-\varepsilon} \left[ix + \ln 2 - \frac{i\pi}{2} + \ln(\sin x)\right] dx$$

$$= \frac{i\pi^2}{2} + \pi \ln 2 - \frac{i\pi^2}{2} + \int_{\varepsilon}^{\pi-\varepsilon} \ln\sin x dx = 0$$

We have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi-\varepsilon} \ln\sin x dx = -\pi \ln 2.$$

Example 4.16 Evaluate $\int_{r_-}^{r_+} dr \frac{\sqrt{r-(r_-)(r_+-r)}}{r}$. Here $r_- < r_+$.

Solution:

There is a branch cut $[r_-, r_+]$. Besides there are two poles r=0 and r=∞. Choose the contour C=C_1+C_2+C_3+C_4+C_5+C_6. C_1=$[r_- + i\varepsilon, r_+ + i\varepsilon]$. C_2:z=$r_+\varepsilon e^{i\theta}$, $-\pi/2 \le \theta\pi/2$. C_3=$[r_+ - i\varepsilon, r_- - i\varepsilon]$. C_4=:z=$r_- + \varepsilon e^{i\theta}$, $-\pi/2 \le \theta\pi/2$. C_5=z=$\rho e^{i\theta}$. C_6: $1/z$=$\rho e^{i\theta}$. We have

$$\oint_C \frac{\sqrt{(z-r_-)(r_+-z)}}{z}dz = 0$$

At the limit $\varepsilon \to 0$ The integration on C_2 and C_4 approach zero. The integration on C_5 is

$$\int_{C_5} \frac{\sqrt{(z-r_-)(r_+-z)}}{z} dz = 2\pi i Res(z=0) = -2\pi\sqrt{r_- r_+}.$$

The integration on C_6 is

$$\int_{C_6} \frac{\sqrt{(z-r_-)(r_+-z)}}{z} dz = \int_{C_5} \frac{-\sqrt{(1-wr_-)(wr_+-1)}}{w^2} dw$$

$$= 2\pi i Res(w=0) = 2\pi i \frac{i}{2}(r_+ + r_-) = -\pi(r_+ + r_-).$$

$$\int_{C_1} \frac{\sqrt{(z-r_-)(r_+-z)}}{z} dz = \int_{r_-}^{r_+} dr \frac{\sqrt{(r-r_-)(r_+-r)}}{r}.$$

$$\int_{C_3} \frac{\sqrt{(z-r_-)(r_+-z)}}{z} dz = -\int_{r_+}^{r_-} dr \frac{\sqrt{(r-r_-)(r_+-r)}}{r} = \int_{r_-}^{r_+} dr \frac{\sqrt{(r-r_-)(r_+-r)}}{r}.$$

Hence we have

$$\int_{r_-}^{r_+} dr \frac{\sqrt{(r-r_-)(r_+-r)}}{r} = \frac{1}{2}(2\pi\sqrt{r_- r_+} + \pi(r_+ + r_-)) = \pi(\sqrt{r_- r_+} + \frac{r_+ + r_-}{2}).$$

IX. PRINCIPAL VALUE

When the integrated function $f(z)$ has a singularity on the curve C the notation $\oint_C f(z)dz$ is actually ill-defined. Nevertheless we can define the "principal value" for such integral. For example,

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x-2} dx.$$

Obviously the point x=2 is a pole. The principal value of such integral is as follows,

$$Pr. \int_{-\infty}^{\infty} \frac{\sin kx}{x-2} dx = \lim_{\varepsilon\to 0} \left(\int_{-\infty}^{2-\varepsilon} \frac{\sin kx}{x-2} dx + \int_{2+\varepsilon}^{\infty} \frac{\sin kx}{x-2} dx \right).$$

Here we learn that the residue calculus is very convenient to calculate such integral. Let us choose the contour as C_+=C_1+C_2+C_3+C_4. Here C_1:$[-R, 2-\varepsilon]$; C_2:z=2+$\varepsilon e^{i\theta}$, $\pi \le \theta \le \pi$. C_3=$[2+\varepsilon, R]$ and C_4:$z = Re^{i\theta}$, $0 \le \theta \le \pi$. Now we have $\oint_C \frac{e^{ikz}}{z-2}dz$=$\int_{C_1} \frac{e^{ikz}}{z-2}dz + \int_{C_2} \frac{e^{ikz}}{z-2}dz + \int_{C_3} \frac{e^{ikz}}{z-2}dz + \int_{C_4} \frac{e^{ikz}}{z-2}dz$=0 since inside and on this contour the function $\frac{e^{ikz}}{z-2}$ is analytic. In particular we evaluate the fourth part approaches zero as R approaches ∞.

$$
\begin{aligned}
& |\int_{C_4} \frac{e^{ikz}}{z-2}dz| = |\int_0^{\pi} \frac{e^{i2k}e^{ikR\cos\theta}e^{-kR\sin\theta}}{Re^{i\theta}-2}Rie^{i\theta}d\theta| \\
\leq & \int_0^{\pi} \frac{|e^{i2k}e^{ikR\cos\theta}e^{-kR\sin\theta}|}{|Re^{i\theta}-2|}|Rie^{i\theta}|d\theta = \int_0^{\pi} \frac{e^{-kR\sin\theta}}{R-2}d\theta \\
= & \ 2 \leq \int_0^{\pi/2} \frac{e^{-kR\sin\theta}}{R-2}d\theta \leq 2 \leq \int_0^{\pi/2} \frac{e^{-2kR\theta/\pi}}{R-2}d\theta \\
= & \ \frac{\pi}{kR(R-2)}[1-e^{-kR}]
\end{aligned}
$$

Next we choose another contour: C_{-}=C_1+C_2+C_3+C_5. Here C_5:z=R $e^{i\theta}$, $\pi \leq \theta \leq 2\pi$. Now we have $\oint_C \frac{e^{-ikz}}{z-2}dz$=$\int_{C_1} \frac{e^{-ikz}}{z-2}dz + \int_{C_2} \frac{e^{-ikz}}{z-2}dz + \int_{C_3} \frac{e^{-ikz}}{z-2}dz + \int_{C_5} \frac{e^{-ikz}}{z-2}dz$=$-2\pi i Res(z=2)$. Here $Res(z=2)$=e^{-2ik}. The minus sign is because this integral is clockwise. We also have the following result for the integration over C_5:

$$
\begin{aligned}
& |\int_{C_5} \frac{e^{-ikz}}{z-2}dz| = |\int_{2\pi}^{\pi} \frac{e^{-i2k}e^{-ikR\cos\theta}e^{kR\sin\theta}}{Re^{i\theta}-2}Rie^{i\theta}d\theta| \\
\leq & \int_{2\pi}^{\pi} \frac{|e^{-i2k}e^{-ikR\cos\theta}e^{kR\sin\theta}|}{|Re^{i\theta}-2|}|Rie^{i\theta}|d\theta = \int_{2\pi}^{\pi} \frac{e^{kR\sin\theta}}{R-2}d\theta \\
= & \leq \int_0^{\pi} \frac{e^{-kR\sin\xi}}{R-2}d\xi \leq 2 \leq \int_0^{\pi/2} \frac{e^{-2kR\xi/\pi}}{R-2}d\xi \\
= & \ \frac{\pi}{kR(R-2)}[1-e^{-kR}]
\end{aligned}
$$

Therefore we know that when $R \to \infty$ and $\varepsilon \to 0$, we have

$$
\begin{aligned}
& \oint_{C_+} \frac{e^{ikz}}{z-2}dz - \oint_{C_-} \frac{e^{ikz}}{z-2}dz = \int_{C_1} \frac{2i\sin kz}{z-2}dz + \int_{C_2} \frac{2i\sin kz}{z-2}dz + \int_{C_2} \frac{2i\sin kz}{z-2}dz \\
= & \ 2iPr. \int_{-\infty}^{\infty} \frac{\sin kx}{x-2}dx + \int_{C_2} \frac{e^{ikz}-e^{-ikz}}{z-2}dz \\
= & \ 2\pi i e^{-2ik}.
\end{aligned}
$$

Here we still need evaluate the integration on C_2 at limit $\varepsilon \to 0$.

$$\int_{C_2} \frac{e^{ikz} - e^{-ikz}}{z-2} dz = \int_{\pi}^{0} \left(\frac{e^{i2k} e^{ik\varepsilon\cos\theta} e^{-k\varepsilon\cos\theta}}{\varepsilon e^{i\theta}} - \frac{e^{-i2k} e^{-ik\varepsilon\cos\theta} e^{k\varepsilon\cos\theta}}{\varepsilon e^{i\theta}} \right) \varepsilon i e^{i\theta} d\theta$$

$$= \int_{\pi}^{0} (e^{i2k} e^{ik\varepsilon\cos\theta} e^{-k\varepsilon\sin\theta} - e^{-i2k} e^{-ik\varepsilon\cos\theta} e^{k\varepsilon\sin\theta}) i d\theta$$

$$\Longrightarrow i(e^{i2k} - e^{-i2k}) \int_{\pi}^{0} d\theta = -\pi i (e^{i2k} - e^{-i2k}) = 2\pi \sin 2k.$$

Therefore

$$2iPr. \int_{-\infty}^{\infty} \frac{\sin kx}{x-2} dx + 2\pi \sin 2k = 2\pi i e^{-2ik}.$$

Hence

$$Pr. \int_{-\infty}^{\infty} \frac{\sin kx}{x-2} dx = \pi \cos 2k.$$

Home Work 4.13 Please evaluate the principal value of the following integral: Pr. $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - a^2} dx$.