Lecture I: Vector Analysis

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(Dated: July 12, 2012)

This lecture introduce the basic elements of vector analysis.

I. VECTORS IN 3-D EUCLIDEAN SPACE

Mathematically any object which has the properties of addition and multiplying a scalar is called "vector". In three-dimensional Euclidean space one can identify an arrow as an "Euclidean vector". The arrow is defined as $\overrightarrow{AB}$ where A is the staring point and B is the ending point. A and B are two point in Euclidean space.

One has the Cartesian coordinates which consists of three unit basis vectors which is normal to each others. A vector is a linear combination of the three basis vectors: $\hat{e}_i, i = 1, 2, 3$ such as

$$\overrightarrow{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3.$$

Naturally one can define the addition as

$$\overrightarrow{F} + \overrightarrow{G} = (F_1 + G_1)\hat{e}_1 + (F_2 + G_2)\hat{e}_2 + (F_3 + G_3)\hat{e}_3.$$

Also one can multiply a vector by a scalar factor:

$$c\overrightarrow{F} = cF_1\hat{e}_1 + cF_2\hat{e}_2 + cF_3\hat{e}_3.$$

The length of a vector $\overrightarrow{F}$ is defined as $|\overrightarrow{F}| = \sqrt{F_1^2 + F_2^2 + F_3^2}$. This is due to the famous Pythagoras theorem. This is the essence of Euclidean space. In this lecture we call "Euclidean vector" as "vector" for short.

II. SCALAR PRODUCT AND VECTOR PRODUCT OF VECTORS IN EUCLIDEAN SPACE

There are two operations on vectors are very important. The first one is scalar product. It is defined as follows:

$$\overrightarrow{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3,\ \overrightarrow{G} = G_1\hat{e}_1 + G_2\hat{e}_2 + G_3\hat{e}_3.$$

The scalar product is defined as

$$\overrightarrow{F} \cdot \overrightarrow{G} = F_1G_1 + F_2G_2 + F_3G_3 = \sum_{i=1,2,3} F_iG_i.$$

The scalar product of two vectors is a scalar. The length of a vector is $|\overrightarrow{F}| = \left(\overrightarrow{F} \cdot \overrightarrow{F}\right)^{1/2}$. Actually one can show the following:

$$\overrightarrow{F} \cdot \overrightarrow{G} = |\overrightarrow{F}||\overrightarrow{G}| \cos\theta.$$

θ is the angle between this two vector.

Another important operation between two vectors are vector product which is defined as

$$\overrightarrow{F} \times \overrightarrow{G} = (F_2G_3 - F_3G_2)\hat{e}_1 + (F_3G_1 - F_1G_3)\hat{e}_2 + (F_1G_2 - F_2G_1)\hat{e}_3.$$

The vector product of two vectors is a vector. To make the definition of the vector product more simple, one can use Levi-Civita symbol: ϵ_{ijk}=1 if ijk is an even permutation of 123. On the other hand ϵ_{ijk}=-1 is ijk is an odd permutation of 123. And ϵ_{ijk}=0 if any two of i, j, k are equal. That is

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1.$$

Others are all zero. With Levi-Civita symbol one can define the vector product as

$$\overrightarrow{F} \times \overrightarrow{G} = \sum_{i,j,k} \epsilon_{ijk}\hat{e}_iF_jG_k.$$

In other words, $(\vec{F} \times \vec{G})_i = \sum_{jk} \epsilon_{ijk}F_jG_k$. From this definition it is trivial to show $\vec{F} \times \vec{G} = -\vec{G} \times \vec{F}$. To save time, we will adopt Einstein notation. Which means, when the same index appears twice, then it means sum over this index from 1 to 3.

Example 1.1: Prove $\vec{F} \times (\vec{G} \times \vec{H}) = \vec{G}(\vec{F} \cdot \vec{H}) - \vec{H}(\vec{F} \cdot \vec{G})$.

Solution:

$$\vec{F} \times (\vec{G} \times \vec{H}) = \epsilon_{ijk}\hat{e}_iF_j(\vec{G} \times \vec{H})_k = \epsilon_{ijk}\hat{e}_iF_j\epsilon_{klm}G_lH_m$$

Here we apply

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

Hence

$$\epsilon_{ijk}\hat{e}_iF_j\epsilon_{klm}G_lH_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\hat{e}_iF_jG_lH_m = \hat{e}_iF_jG_iH_j - \hat{e}_iF_lG_lH_i = \vec{G}(\vec{F} \cdot \vec{H}) - \vec{H}(\vec{F} \cdot \vec{G}).$$

Example 1.2: Prove $\vec{F}\cdot\vec{G}\times\vec{H}=\vec{G}\cdot\vec{H}\times\vec{F}=\vec{H}\cdot\vec{F}\times\vec{G}$

Solution:

$$\vec{F}\cdot\vec{G}\times\vec{H}=F_i(\vec{G}\times\vec{H})_i=F_i\epsilon_{ijk}G_jH_k=\epsilon_{ijk}F_iG_jH_k.$$

Similar $\vec{G}\cdot\vec{H}\times\vec{F}=\epsilon_{ijk}G_iH_jF_k$ and $\vec{H}\cdot\vec{F}\times\vec{G}=\epsilon_{ijk}H_iF_jG_k$. From the property of Livi-Civeta symbol we have

$$\epsilon_{ijk}F_iG_jH_k=\epsilon_{jki}F_iG_jH_k=\epsilon_{lmn}F_nG_lH_m=\epsilon_{lmn}G_lH_mF_n=\vec{G}\cdot(\vec{H}\times\vec{F}).$$

Similarly one can prove it is identical to $\vec{H}\cdot\vec{F}\times\vec{G}$.

Exercise 1.1: Please show that $(\overrightarrow{F}\cdot\overrightarrow{G})^2+|\overrightarrow{F}\times\overrightarrow{G}|^2=|\overrightarrow{F}|^2|\overrightarrow{G}|^2$.

Solution:

$$\begin{aligned}
|\overrightarrow{F}\times\overrightarrow{G}|^2 &= (\epsilon_{ijk}F_jG_k)(\epsilon_{ilm}F_lG_m)=(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})F_jG_kF_lG_m\\
&= F_jF_jG_kG_k-F_jG_jG_kF_k=(\overrightarrow{F}\cdot\overrightarrow{F})(\overrightarrow{G}\cdot\overrightarrow{G})-(\overrightarrow{F}\cdot\overrightarrow{G})(\overrightarrow{F}\cdot\overrightarrow{G})\\
(\overrightarrow{F}\cdot\overrightarrow{G})^2 &+ |\overrightarrow{F}\times\overrightarrow{G}|^2=(\overrightarrow{F}\cdot\overrightarrow{F})(\overrightarrow{G}\cdot\overrightarrow{G})=|\overrightarrow{F}|^2|\overrightarrow{G}|^2.
\end{aligned}$$

From this result one can derive that $|\vec{F}\times\vec{G}|=|\vec{F}||\vec{G}|\sin\theta$.

Home Work 1.1: Prove that $(\overrightarrow{A}\times\overrightarrow{B})\cdot(\overrightarrow{C}\times\overrightarrow{D})=(\overrightarrow{A}\cdot\overrightarrow{C})(\overrightarrow{B}\cdot\overrightarrow{D})-(\overrightarrow{A}\cdot\overrightarrow{D})(\overrightarrow{B}\cdot\overrightarrow{C})$.

Home Work 1.2: Prove that $\overrightarrow{F}\times(\overrightarrow{G}\times\overrightarrow{H})+\overrightarrow{G}\times(\overrightarrow{H}\times\overrightarrow{F})+\overrightarrow{H}\times(\overrightarrow{F}\times\overrightarrow{G})=0$.

III. GRADIENT, CURL AND DIVERGENCE

The basic elements of vector analysis are the following operations:

1. Gradient: $\nabla f=\frac{\partial f}{\partial x^1}\hat{e}_1+\frac{\partial f}{\partial x_2}\hat{e}_2+\frac{\partial f}{\partial x_3}\hat{e}_3$. It operates on a scalar function and obtains a vector.

2. Curl: $\nabla\times\overrightarrow{F}=\epsilon_{ijk}\hat{e}_i\frac{\partial F_k}{\partial x^j}$. It operates on a vectorial function and obtains another vectorial function.

3. Divergence: $\nabla\cdot\overrightarrow{F}=\frac{\partial F_i}{\partial x^i}$. It operates on an vectorial function and obtains a scalar function.

4. Laplacian: $\nabla^2 f=\nabla\cdot\nabla f$. It operates on a scalar function and obtains a scalar function.

One can combine those operations and make rather complicated operations.

Example 1.3: Show that (a) $\nabla\times\nabla f(x)$=0. (b) $\nabla\cdot\nabla\times\vec{F}(x)$=0. Here f is a scalar function and $\vec{F}(x)$ is a vector.

Solution:

(a)

$$\nabla\times\nabla f(x)=\epsilon_{ijk}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}f(x)=\epsilon_{ijk}\frac{\partial^2 f(x)}{\partial x^j\partial x^k}=\epsilon_{ikj}\frac{\partial^2 f(x)}{\partial x^k\partial x^j}=-\nabla\times\nabla f(x).$$

Hence $\nabla\times\nabla f(x)$=0

(b)

$$\nabla\cdot\nabla\times\vec{F}(x)=\frac{\partial}{\partial x^i}\epsilon_{ijk}\frac{\partial F_k(x)}{\partial x^j}=\epsilon_{ijk}\frac{\partial^2 F_k(x)}{\partial x^i\partial x^j}=\epsilon_{jik}\frac{\partial^2 F_k(x)}{\partial x^j\partial x^i}=-\epsilon_{ijk}\frac{\partial^2 F_k(x)}{\partial x^i\partial x^j}=-\nabla\cdot\nabla\times\vec{F}(x)$$

Hence $\nabla\cdot\nabla\times\vec{F}(x)$=0.

Example 1.4:

(a)Show that $\nabla\times(\vec{F}\times\vec{G})$=$\vec{F}(\nabla\cdot\vec{G})-(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}-\vec{G}(\nabla\cdot\vec{F})$.

(b)$\nabla\times(\nabla\times\vec{F})$=$\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$.

(c)$\nabla(\vec{F}\cdot\vec{G})$=$(\vec{F}\cdot\nabla)G+(\vec{G}\cdot\nabla)F+\vec{F}\times\nabla\times\vec{G}+\vec{G}\times\nabla\times\vec{F}$

Solution:

(a)

$$\nabla\times(\vec{F}\times\vec{G})=\epsilon_{abi}\frac{\partial(\vec{F}\times\vec{G})_i}{\partial x^b}\hat{e}_a=\epsilon_{abi}\epsilon_{ijk}\frac{\partial}{\partial x^b}(F^jG^k)\hat{e}_a.$$

Because $\epsilon_{abi}\epsilon_{ijk}=\epsilon_{iab}\epsilon_{ijk}=\delta_{aj}\delta_{bk}-\delta_{ak}\delta_{bj}$. Hence

$$\begin{aligned}[\nabla\times(\vec{F}\times\vec{G})]_a &= (\delta_{aj}\delta_{bk}-\delta_{ak}\delta_{bj})\frac{\partial}{\partial x^b}(F^jG^k)=\frac{\partial}{\partial x^b}(F^aG^b)-\frac{\partial}{\partial x^b}(F^bG^a)\\ &= \frac{\partial F^a}{\partial x^b}G^b+\frac{\partial G^b}{\partial x^b}F^a-\frac{\partial F^b}{\partial x^b}G^a-\frac{\partial G^a}{\partial x^b}F^b\end{aligned}$$

$$\nabla\times(\vec{F}\times\vec{G})=\vec{G}\cdot\nabla\vec{F}+(\nabla\cdot\vec{G})\vec{F}-(\nabla\cdot\vec{F})\vec{G}-\vec{F}\cdot\nabla\vec{G}$$

(b)

$$\begin{aligned}[\nabla\times(\nabla\vec{F})]_a &= \epsilon_{abi}\frac{\partial}{\partial x^b}\epsilon_{ijk}\frac{\partial F^k}{\partial x^j}=(\delta_{aj}\delta_{bk}-\delta_{ak}\delta_{bj})\frac{\partial F^k}{\partial x^b\partial x^j}\\ &= \frac{\partial^2F^b}{\partial x^b\partial x^a}-\frac{\partial^2 F^a}{\partial x^{b2}}=[\nabla(\nabla\cdot\vec{F})]_a-[\nabla^2\vec{F}]_a\end{aligned}$$

(c)

$$\vec{F} \times (\nabla \times \vec{G})_a = \epsilon_{abi}\epsilon_{ijk} F^b \frac{\partial G^k}{\partial x^j} = (\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) F^b \frac{\partial G^k}{\partial x^j}$$
$$= F^b \frac{\partial G^b}{\partial x^a} - F^b \frac{\partial G^a}{\partial x^b}$$

$$\vec{G} \times (\nabla \times \vec{F})_a = G^b \frac{\partial F^b}{\partial x^a} - G^b \frac{\partial F^a}{\partial x^b}.$$

Hence

$$\vec{F} \times (\nabla \times \vec{G})_a + \vec{G} \times (\nabla \times \vec{F})_a = F^b \frac{\partial G^b}{\partial x^a} - F^b \frac{\partial G^a}{\partial x^b} + G^b \frac{\partial F^b}{\partial x^a} - G^b \frac{\partial F^a}{\partial x^b}$$
$$= \frac{\partial (F^b G^b)}{\partial x^a} - (\vec{F} \cdot \nabla)\vec{G}_a - (\vec{G} \cdot \nabla)\vec{F}_a$$

$$\vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) = \nabla(\vec{F} \cdot \vec{G}) - (\vec{F} \cdot \nabla)\vec{G} - (\vec{G} \cdot \nabla)\vec{F}$$

Home Work 1.3:

(a) Show that $\nabla \cdot (\vec{F} \times \vec{G})$=$\vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$.

(b)Show that $\nabla \times (f\nabla f)$=0.

IV. STOKE'S THEOREM AND GAUSS THEOREM

The basic theorems of vector analysis are the following:

1. Divergence Theorem:
$$\oint_S \overrightarrow{F} \cdot \overrightarrow{n} dA = \int_V \nabla \cdot \overrightarrow{F} dV.$$
Here S is the boundary of V. $\overrightarrow{n}$ is the unit normal vector of the surface boundary.

2. Stokes' Theorem:
$$\oint_C \overrightarrow{F} \cdot d\vec{l} = \int_S \nabla \times \overrightarrow{F} \cdot \overrightarrow{n} dA.$$
Here C is the boundary of S. $\overrightarrow{n}$ is the unit normal vector of the surface. $d\vec{l}$ is the unit tangent vector of the boundary curve.

To define $\hat{n}$, if the surface is defined as $\vec{r}=\vec{r}(u,v)$. Then

$$\vec{N}=\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v},\ \vec{n}=\frac{\vec{N}}{|\vec{N}|}.$$

The surface area element is defined as

$$d\vec{A}=\vec{N}dudv,\ dA=|\vec{N}|dudv.$$

In fact, using more advanced mathematical language, these two theorems are one, it is

$$\int_{\Omega}d\omega=\int_{\partial\Omega}\omega.$$

This language is called "differential form". Interesting readers should find it in the standard textbooks.

Example 1.5 Prove that $\oint_C \Phi d\vec{l}=\int_S \hat{n}\times\nabla\Phi dA$.

Solution: From Stokes theorem $\oint_C \overrightarrow{F}\cdot d\vec{l}=\int_S \nabla\times\overrightarrow{F}\cdot\overrightarrow{n}dA$, Choose $\overrightarrow{F}=\Phi\overrightarrow{K}$. Here $\overrightarrow{K}$ is an arbitrary nonzero vector. Then

$$\oint_C \Phi\overrightarrow{K}\cdot d\vec{l}=\int_S \nabla\times\Phi\overrightarrow{K}\cdot\overrightarrow{n}dA=\int_S \nabla\Phi\times\overrightarrow{K}\cdot\overrightarrow{n}dA=\int_S \overrightarrow{K}\cdot\overrightarrow{n}\times\nabla\Phi dA.$$
$$\Longrightarrow \overrightarrow{K}\cdot\left(\oint_C \Phi d\vec{l}-\int_S \overrightarrow{n}\times\nabla\Phi dA\right)=0.$$

Hence $\oint_C \Phi d\vec{l}=\int_S \overrightarrow{n}\times\nabla\Phi dA$.

Example 1.6: Please prove that $\oint_S \hat{n}\times\overrightarrow{G}dA=\int_V \nabla\times\overrightarrow{G}dV$

Solution:

From Divergence theorem: $\oint_S \overrightarrow{F}\cdot\overrightarrow{n}dA=\int_V \nabla\cdot\overrightarrow{F}dV$, we choose $\overrightarrow{F}=\overrightarrow{K}\times\overrightarrow{G}$ here $\overrightarrow{K}$ is an arbitrary constant nonzero vector. Then

$$\begin{aligned}\oint_S \overrightarrow{F}\cdot\overrightarrow{n}dA &= \oint_S \overrightarrow{K}\times\overrightarrow{G}\cdot\overrightarrow{n}dA=\oint_S \overrightarrow{K}\cdot\overrightarrow{G}\times\hat{n}dA\\ \int_V \nabla\cdot\overrightarrow{G}dV &= \int_V \nabla\cdot(\overrightarrow{K}\times\overrightarrow{G})dV=-\int_V \overrightarrow{K}\cdot(\nabla\times\overrightarrow{G})dV\\ &\Longrightarrow \overrightarrow{K}\cdot\left(\oint_S \overrightarrow{G}\times\hat{n}dA+\int_V(\nabla\times\overrightarrow{G})dV\right)=0.\\ &\Longrightarrow \oint_S \overrightarrow{G}\times\hat{n}dA+\int_V(\nabla\times\overrightarrow{G})dV=0.\end{aligned}$$

Hence $\oint_S \hat{n} \times \overrightarrow{G} dA = \int_V \nabla \times \overrightarrow{G} dV$. Here we apply

$$\begin{aligned}\nabla \cdot (\overrightarrow{K} \times \overrightarrow{G}) &= \frac{\partial}{\partial x^i}\epsilon_{ijk}K_j G_k = \epsilon_{ijk}K_j \frac{\partial G_k}{\partial x^i} \\ &= -K_j \epsilon_{jik}\frac{\partial G_k}{\partial x^i} = -\overrightarrow{K} \cdot \nabla \times \overrightarrow{G}.\end{aligned}$$

Example: 1.7: Prove that if $\nabla \times \vec{F} = 0$ then there exists φ such that $\vec{F} = -\nabla\varphi$.

Solution:

Choose two different pathes from the origin O to any point A. The first path is called P_1 and the second path is called P_2. The area enclosed by P_1 and P_2 is called S. Since $\nabla \times \vec{F} = 0, \Longrightarrow \int_S \nabla \times \vec{F} = 0$ From Stock's theorem: $\int_C \vec{F} \cdot d\vec{l}$=0. C is the boundary of S which is $P_1 - P_2$. Hence $\int_{P_1} \vec{F} \cdot d\vec{l} - \int_{P_2} \vec{F} \cdot d\vec{l} = 0$. It means the value of $\int_P \vec{F} \cdot d\vec{l}$ is independent of the path. P is any path connecting O and A. For every point C in the whole space, Define $\varphi(\vec{r} = \overrightarrow{OC}) = \int_P \vec{F} \cdot d\vec{l}$ here P is any path connecting the origin O and C. Now taking gradient of φ:

$$-\nabla\varphi = \nabla \int_0^C \vec{F} \cdot d\vec{l} = \int_C^{C+\delta C} \vec{F} \cdot d\vec{l} = \vec{F}(\vec{r} = \overrightarrow{OC}).$$

Exercise 1.2: Prove that (a) $\int_V \nabla\Phi dV$=$\oint_S \Phi\hat{n}dA$. (b) $\int_S(\hat{n} \times \nabla) \times \overrightarrow{G}dA$=$\oint_C d\vec{l} \times \overrightarrow{G}$.

Home Work 1.4: Prove that $\int_V(f\nabla^2 g - g\nabla^2 f)dV$=$\oint_S(f\nabla g - g\nabla f) \cdot \hat{n}dA$.

Home Work 1.5: If C is a close loop in the plane. Please evaluate $\oint_C \overrightarrow{r} \times d\vec{l}$. Assume the origin is inside the loop.

V. DELTA FUNCTION

Here I would like to introduce Dirac delta function which is very useful object in this course. First, let us calculate the following quantity:

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = \nabla\cdot\left(\frac{-(x_1-x_1')\hat{e}_1+(x_2-x_2')\hat{e}_2+(x_3-x_3')\hat{e}_3}{(\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2})^3}\right)$$
$$= \frac{-1}{(\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2})^3} + \left(\frac{3(x_1-x_1')^2}{(\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2})^5}\right)$$
$$+ \frac{-1}{(\sqrt{(x_2-x_2')^2+(x_2-x_2')^2+(x_3-x_3')^2})^3} + \left(\frac{3(x_2-x_2')^2}{(\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2})^5}\right)$$
$$+ \frac{-1}{(\sqrt{(x_3-x_3')^2+(x_2-x_2')^2+(x_3-x_3')^2})^3} + \left(\frac{3(x_3-x_3')^2}{(\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2})^5}\right)$$
$$= \frac{-3}{|\vec{r}-\vec{r'}|^3} + 3\frac{|\vec{r}-\vec{r'}|^2}{|\vec{r}-\vec{r'}|^5} = 0.$$

This is hold when $\vec{r} \neq \vec{r'}$. However when $\vec{r}$ is close to $\vec{r'}$ we find something interesting. Assume B is the ball with the radius ρ and the center at A here $\overrightarrow{OA}$=$\vec{r'}$. We choose ρ to be very small. Then by calculating the following integral:

$$\int_B \nabla\cdot\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)d^3V = \int_S \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\cdot\hat{n}dA = -\int_S \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\cdot\hat{n}dA.$$
$$= -\int_S \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|}dA = -\int_S \frac{|\vec{r}-\vec{r'}|^2}{|\vec{r}-\vec{r'}|^4}dA$$
$$= -\int_S \frac{1}{\rho^2}dA = -\int \frac{1}{\rho^2}\rho^2 d\Omega = -\int d\Omega = -4\pi.$$

Actually we can even prove that

$$\begin{aligned}
&\int_B \left(\nabla\cdot\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right) f(\vec{r})d^3V \\
=& \int_B \nabla\cdot\left(f(\vec{r})\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right)d^3V - \int_B \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\cdot\nabla f(\vec{r})d^3V \\
=& \int_S f(\vec{r})\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\cdot\hat{n}dA - \int_B \frac{\vec{r'}-\vec{r}}{|\vec{r}-\vec{r'}|^3}\cdot\nabla f(\vec{r})d^3V \\
=& \int_S \left(f(\vec{r})\frac{-\vec{r}+\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)\cdot\hat{n}dA + \int_V \frac{1}{r^2}\frac{\partial f}{\partial r}r^2 drd\Omega \\
=& -\int_S f(\vec{r})d\Omega + \int_0^\rho \frac{\partial f}{\partial r}drd\Omega = -\int_S f(\vec{r})d\Omega + \int (f(\vec{r})|_{|\vec{r}-\vec{r'}|=\rho} - f(\vec{r})|_{|\vec{r}-\vec{r'}|=0})d\Omega = -4\pi f(\vec{r'}).
\end{aligned}$$

Here r=$|\vec{r}-\vec{r'}|$ and dA=$r^2d\Omega$. Hence we know

$$\nabla^2\frac{1}{|\vec{r}-\vec{r'}|} = -4\pi\delta(\vec{r}-\vec{r'}) = -4\pi\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3').$$

here $\delta(x_1 - x;_1)$ is called Dirac delta function which is defined as

$$\int_a^b f(x)\delta(x-x_0) = f(x_0), if x_0 \in [a,b],\ \int_a^b f(x)\delta(x-x_0) = 0, otherwise.$$

Obviously we know $\delta(x-x_0)$=0 when $x \neq 0$. However the value of the function at $x = x_0$ is not well defined. Therefore one must take Dirac delta function as the limit function of a sequence of functions. Namely there should be a sequence $f_n(x)$ and when $n \to \infty$, $f_n(x) \to \delta(x-x_0)$. Several properties of this function can be derived.

Example 1.8: Prove that

(a):$\int_{-\infty}^{\infty} f(x)\delta(a(x-x_0))dx$=$\frac{1}{a}f(x_0)$. (b):Evaluate $\int \delta(x^2-a^2)f(x)dx$.

Solution:

(a):

$$\int_{-\infty}^{\infty} f(x)\delta(a(x-x_0))dx = \int_{-\infty}^{\infty} f(y/a)\delta(y-ax_0)d\frac{y}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y-ax_0)dy = \frac{1}{a}f(x_0).$$

(b):

$$\int_{-\infty}^{\infty} f(x)\delta(x^2-a^2)dx = \int_0^{\infty} f(\sqrt{y})\delta(y-a^2)d\frac{y}{2\sqrt{y}} + \int_0^{\infty} f(-\sqrt{y})\delta(y-a^2)d\frac{y}{-2\sqrt{y}} = \frac{f(|a|)}{2|a|} + \frac{f(-|a|)}{-2|a|}.$$

Exercise 1.3: Show that $\int \delta'(x-x_0)f(x)dx$=$-f'(x_0)$.

VI. HELMHOLTZ'S THEOREM

Here we will apply the previous section to derive the following theorem. It is important for electromagnetism and fluid Mechanics.

If $\nabla \cdot \vec{F}(\vec{r})$=$s(\vec{r})$ and $\nabla \times \vec{F}(\vec{r})$=$\vec{c}(\vec{r})$ then $\vec{F}(\vec{r})$=$-\nabla\varphi(\vec{r}) + \nabla \times \vec{\psi}(\vec{r})$ where

$$\varphi(\vec{r}) = \frac{1}{4\pi}\int d^3r' \frac{s(\vec{r'})}{|\vec{r}-\vec{r'}|}, \quad \vec{\psi} = \frac{1}{4\pi}\int d^3r' \frac{\vec{c}(\vec{r'})}{|\vec{r}-\vec{r'}|}.$$

Here it is assumed that $\vec{c}(\vec{r'})$ approaches zero faster $\frac{1}{|\vec{r'}|}$.

Proof:

It is straightforward proof. We simply operate divergence and curl on the object constructed. First, let us take the divergence of $\vec{F}$:

$$\begin{aligned} & \nabla \cdot \vec{F} = -\nabla \cdot \nabla\varphi + \nabla \cdot \nabla \times \vec{\psi} = -\nabla^2\varphi(\vec{r}) \\ = & -\nabla^2 \frac{1}{4\pi}\int d^3r' \frac{s(\vec{r'})}{|\vec{r}-\vec{r'}|} \; = \frac{-1}{4\pi}\int d^3r' \nabla^2 \left(\frac{s(\vec{r'})}{|\vec{r}-\vec{r'}|}\right) \\ = & \frac{-1}{4\pi}\int d^3r' s(\vec{r'}) \nabla^2 \left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = \frac{-1}{4\pi}\int d^3r' s(\vec{r'})(-4\pi)\delta^3(\vec{r}-\vec{r'}) \\ = & \; s(\vec{r}) \end{aligned}$$

Indeed it gives the right answer. next step is to take the curl of the expression:

$$\nabla \times \vec{F} = -\nabla \times \nabla\varphi + \nabla \times (\nabla \times \vec{\psi}) = \nabla(\nabla \cdot \vec{\psi}(\vec{r})) - \nabla^2\vec{\psi}(\vec{r}).$$

The first term is given as

$$\begin{aligned} & \nabla(\nabla \cdot \vec{\psi}(\vec{r})) = \nabla\left(\nabla \cdot \frac{1}{4\pi}\int dV \frac{\vec{c}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right) = \nabla \int d^3r' \vec{c}(\vec{r'}) \cdot \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) \\ = & \frac{1}{4\pi}\int d^3r' c_i(\vec{r'}) \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\frac{1}{|\vec{r}-\vec{r'}|}\hat{e}_k = \frac{1}{4\pi}\int d^3r' c_i(\vec{r'}) \frac{\partial^2}{\partial x_k \partial x_i}\left[\frac{1}{|\vec{r}-\vec{r'}|}\right]\hat{e}_k \end{aligned}$$

Then we notice that $\frac{\partial^2}{\partial x_k \partial x_i}\left[\frac{1}{|\vec{r}-\vec{r'}|}\right] = \frac{\partial^2}{\partial x'_k \partial x'_i}\left[\frac{1}{|\vec{r}-\vec{r'}|}\right]$ hence we have

$$\nabla(\nabla\cdot\vec{\psi}(\vec{r})) = \frac{1}{4\pi}\int d^3r' c_i(\vec{r'})\frac{\partial^2}{\partial x'_k \partial x'_i}\left[\frac{1}{|\vec{r}-\vec{r'}|}\right]\hat{e}_k$$
$$= \frac{1}{4\pi}\int d^3r'\frac{\partial}{\partial x'_i}\left[c_i(\vec{r'})\frac{\partial}{\partial x'_k}\frac{1}{|\vec{r}-\vec{r'}|}\right]\hat{e}_k - \frac{1}{4\pi}\int d^3r'\frac{\partial c_i(\vec{r'})}{\partial x'_i}\frac{\partial}{\partial x'_k}\left[\frac{1}{|\vec{r}-\vec{r'}|}\right]\hat{e}_k$$

But $\nabla'\cdot\vec{c}(\vec{r'}) = 0$ since $\vec{c}(\vec{r'}) = \nabla'\times\vec{F}(\vec{r'})$. Therefore the second term vanishes.

$$\nabla(\nabla\cdot\vec{\psi}(\vec{r}))_k = \frac{1}{4\pi}\int d^3r'\nabla\cdot\left(\vec{c}(\vec{r'})\left[\frac{\partial}{\partial x'_k}\frac{1}{|\vec{r}-\vec{r'}|}\right]\right)$$
$$= \frac{1}{4\pi}\int_S \vec{c}(\vec{r'})\left[\frac{\partial}{\partial x'_k}\frac{1}{|\vec{r}-\vec{r'}|}\right]\cdot\hat{n}dA' \Longrightarrow 0.$$

If $\vec{c}(\vec{r'})$ approaches zero faster $\frac{1}{|\vec{r'}|}$. Note that $\nabla'\cdot\vec{C}(\vec{r'}) = 0$. The second term is given as

$$-\nabla^2\vec{\psi} = \frac{-1}{4\pi}\nabla^2\int dV\left(\frac{\vec{c}(\vec{r})}{|\vec{r}-\vec{r'}|}\right) = \frac{-1}{4\pi}\int d^3r' c_i(\vec{r'})\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\hat{e}_i$$
$$= \frac{-1}{4\pi}\int d^3r' c_i(\vec{r'})(-4\pi)\delta^3(\vec{r}-\vec{r'})\hat{e}_i = c_i(\vec{r})\hat{e}_i = \vec{c}(\vec{r}).$$
$$\Longrightarrow \nabla\cdot\vec{F} = s(\vec{r}),\ \ \nabla\times\vec{F} = \vec{c}(\vec{r'}).$$

Example 1.9 prove that $\vec{G} = 0$ if $\nabla\cdot\vec{G} = \nabla\times\vec{G} = 0$.

Solution:

Since $\nabla\times\vec{G}$=0 we have $\vec{G}$=$-\nabla\varphi$.

$$0 = \int\varphi\nabla^2\varphi dV = \int\nabla\cdot(\varphi\nabla\varphi)dV - \int(\nabla\varphi)\cdot(\nabla\cdot\varphi)dV = \int_S\varphi\nabla\varphi\cdot\hat{n}dA - \int(\nabla\varphi)\cdot(\nabla\cdot\varphi)dV$$
$$S\to\infty \Longrightarrow \int(\nabla\varphi)\cdot(\nabla\varphi)dV = \int\vec{G}\cdot\vec{G}dV = 0.$$

Hence $\vec{G}$=0.

VII. CURVILINEAR COORDINATES

Sometimes Cartesian coordinate is not very convenient and we need use other coordinates. To identify a point one needs three coordinate functions: $q^i(x^1, x^2, x^3), i = 1, 2, 3$. There are two way to generate the basis vectors. The first way is to fix two coordinate functions. By this way one obtains a curve with one parameters. The tangent vector of this curve is identified as one basic vector. In other words we can define the basis vectors as

$$\vec{b}_i = \frac{\partial x^1}{\partial q^i}\hat{e}_1 + \frac{\partial x^2}{\partial q^i}\hat{e}_2 + \frac{\partial x^3}{\partial q^i}\hat{e}_3 = \frac{\partial \overrightarrow{r}}{\partial q^i}.$$

On the other hand, there is another way. Namely one can choose the gradients of the coordinate functions to be the basis vectors.

$$\vec{b}^i = \frac{\partial q^i}{\partial x^1}\hat{e}_1 + \frac{\partial q^i}{\partial x^2}\hat{e}_2 + \frac{\partial q^i}{\partial x^3}\hat{e}_3 = \nabla q^i.$$

Interestingly one has

$$\vec{b}^i \cdot \vec{b}_j = \delta^i_j.$$

Therefore a vector $\overrightarrow{F}$ can be expressed as

$$\overrightarrow{F} = \tilde{F}^1\vec{b}_1 + \tilde{F}^2\vec{b}_2 + \tilde{F}^3\vec{b}_3 = \tilde{F}_1\vec{b}^1 + \tilde{F}_2\vec{b}^2 + \tilde{F}_3\vec{b}^3.$$

How to relate $\tilde{F}^i$ and $\tilde{F}_i$? The crucial point is to define the following quantity:

$$ds^2 = d\vec{r}\cdot d\vec{r} = \frac{\partial \overrightarrow{r}}{\partial q^i}dq^i \cdot \frac{\partial \overrightarrow{r}}{\partial q_j}dq^j = \tilde{g}_{ij}dq^i dq^j.$$

Here

$$\tilde{g}_{ij} = \frac{\partial x^m}{\partial q^i}\frac{\partial x^m}{\partial q^j}.$$

Then one has

$$\tilde{g}_{ij}\vec{b}^j = \frac{\partial x^k}{\partial q^i}\frac{\partial x^k}{\partial q^j}\frac{\partial q^j}{\partial x^l}\hat{e}_l = \frac{\partial x^k}{\partial q^i}\delta_{kl}\hat{e}_l = \frac{\partial x^k}{\partial q^i}\hat{e}_k = \vec{b}_i.$$

Apply this relation one has

$$\begin{aligned}
\tilde{F}_1 &= \tilde{F}^1\tilde{g}_{11} + \tilde{F}^2\tilde{g}_{21} + \tilde{F}^3\tilde{g}_{31},\\
\tilde{F}_2 &= \tilde{F}^1\tilde{g}_{12} + \tilde{F}^2\tilde{g}_{22} + \tilde{F}^3\tilde{g}_{32},\\
\tilde{F}_3 &= \tilde{F}^1\tilde{g}_{13} + \tilde{F}^2\tilde{g}_{23} + \tilde{F}^3\tilde{g}_{33},
\end{aligned}$$

That is

$$\tilde{F}_k = \tilde{g}_{kl}\tilde{F}^l.$$

To invert this relation, we define the following quantity:

$$\tilde{g}^{ij}\tilde{g}_{jk} = \delta^i_k.$$

One should find that

$$\tilde{g}^{ij}\tilde{F}_j = \tilde{g}^{ij}\tilde{g}_{jk}\tilde{F}^k = \tilde{F}^i.$$

The upper and lower index are only convenient but really own very concrete meaning. It can only be seen by transforming the coordinates. Suppose now we have another coordinate functions p_i. Similarly we can define

$$\vec{c}_i = \frac{\partial \overrightarrow{r}}{\partial p^i}, \ \vec{c}^i = \nabla p^i.$$

The relationships between two coordinates are as follow:

$$\begin{aligned}
\vec{c}_i &= \frac{\partial \overrightarrow{r}}{\partial p^i} = \frac{\partial \overrightarrow{r}}{\partial q^j}\frac{\partial q^j}{\partial p^i} = \vec{b}_j\frac{\partial q^j}{\partial p^i}. \\
\vec{c}^i &= \frac{\partial p^i}{\partial x^k}\hat{e}_k = \frac{\partial p^i}{\partial q^j}\frac{\partial q^j}{\partial x^k}\hat{e}_k = \frac{\partial p^i}{\partial q^j}\overrightarrow{c}^j.
\end{aligned}$$

A vector $\overrightarrow{F}$ can express as

$$\overrightarrow{F} = \bar{F}^1\vec{c}_1 + \bar{F}^2\vec{c}_2 + \bar{F}^3\vec{c}_3 = \tilde{F}^1\vec{b}_1 + \tilde{F}^2\vec{b}_2 + \tilde{F}^3\vec{b}_3.$$

It is easy to see

$$\begin{aligned}
\tilde{F}^1 &= \bar{F}^1\frac{\partial q^1}{\partial p^1} + \bar{F}^2\frac{\partial q^1}{\partial p^2} + \bar{F}^3\frac{\partial q^1}{\partial p^3}. \\
\tilde{F}^2 &= \bar{F}^1\frac{\partial q^2}{\partial p^1} + \bar{F}^2\frac{\partial q^2}{\partial p^2} + \bar{F}^3\frac{\partial q^2}{\partial p^3}. \\
\tilde{F}^3 &= \bar{F}^1\frac{\partial q^3}{\partial p^1} + \bar{F}^2\frac{\partial q^3}{\partial p^1} + \bar{F}^3\frac{\partial q^3}{\partial p^3}.
\end{aligned}$$

That is

$$\tilde{F}^k = \bar{F}^l\frac{\partial q^k}{\partial p^l}.$$

A quantity transforms in this way is called contravariant vector. Similarly we have

$$\overrightarrow{F} = \bar{F}_1\vec{c}^1 + \bar{F}_2\vec{c}^2 + \bar{F}_3\vec{c}^3 = \bar{F}_1\vec{b}^1 + \bar{F}_2\vec{b}^2 + \bar{F}_3\vec{b}^3.$$

And one has

$$\tilde{F}_k = \bar{F}_l \frac{\partial p^l}{\partial q^k}.$$

A quantity transforms in this way is called covariant vector.

Let us remember

$$ds^2 = d\vec{r} \cdot d\vec{r} = \frac{\partial \overrightarrow{r}}{\partial p^i} dp^i \cdot \frac{\partial \overrightarrow{r}}{\partial p^j} dp^j = \bar{g}_{ij} dp^i dp^j.$$

Here

$$\bar{g}_{ij} = \frac{\partial x^k}{\partial p^i} \frac{\partial x^k}{\partial p^j}.$$

Again we have

$$\bar{F}_k = \bar{g}_{kl} \bar{F}^l.$$

Since

$$\begin{aligned} \tilde{F}_k &= \bar{F}_l \frac{\partial p^l}{\partial q^k} = \bar{g}_{lj} \bar{F}^j \frac{\partial p^l}{\partial q^k} = \bar{g}_{lj} \frac{\partial p^l}{\partial q^k} \frac{\partial p^j}{\partial q^i} \tilde{F}^i \\ &= \tilde{g}_{ki} \tilde{F}^i. \end{aligned}$$

So one has

$$\bar{g}_{lj} \frac{\partial p^l}{\partial q^k} \frac{\partial p^j}{\partial q^i} \tilde{F}^i = \tilde{g}_{ki} \tilde{F}^i.$$

This is consistent with the definition since

$$\begin{aligned} \tilde{g}_{ki} &= \frac{\partial x^m}{\partial q^k} \frac{\partial x^m}{\partial q^i} = \frac{\partial x^m}{\partial p^l} \frac{\partial p^l}{\partial q^k} \frac{\partial x^m}{\partial p^j} \frac{\partial p^j}{\partial q^i} \\ &= \bar{g}_{lj} \frac{\partial p^l}{\partial q^k} \frac{\partial p^j}{\partial q^i} \end{aligned}$$

Therefore g_{ij} is a covariant quantity, but it is not a vector but a tensor. It is called metric tensor.

Home Work 1.6 Please obtain the $\vec{b}_i$ and $\vec{b}^i$ and g_{ij} for the following coordinates:

(a) Spherical coordinate: $x_1 = r \sin\theta \cos\phi$, $x_2 = r \sin\theta \sin\phi$ and $x_3 = r \cos\theta$.

(b) Cylindrical coordinate: $x_1 = \rho \cos\theta$, $x_2 = \rho \sin\theta$ and $x_3 = z$.

(c) Parabolic cylindrical coordinate: $x_1 = \sigma\tau$ and $x_2 = \frac{\tau^2 - \sigma^2}{2}$, $x_3 = z$.

(d) Parabolic coordinate: $x_1 = \sigma\tau \cos\varphi$ and $x_2 = \sigma\tau \sin\varphi$ and $x_3 = \frac{1}{2}(\sigma^2 - \tau^2)$.

VIII. ORTHOGONAL COORDINATES IN EUCLIDEAN SPACE

Here we will restrict ourselves to discuss the coordinate whose basis vectors are orthogonal to each other. In other words, $\vec{b}_i \cdot \vec{b}_j$=0 if $i \neq j$. Naturally we have

$$g_{ij} = \vec{b}_i \cdot \vec{b}_j = h_i h_2 \delta_{ij}.$$

Here h^i=$|\vec{b}_i| = |\frac{\partial \vec{r}}{\partial q^i}|$. Consequently one has

$$d\vec{r} = \vec{b}_i dq^i = h_i dq^i \hat{b}_i.$$

Here $\hat{b}_i = \frac{\vec{b}_i}{|\vec{b}_i|}$. Furthermore for a surface area element

$$\hat{n}_i dA = \vec{N}_i = \epsilon_{ijk} h_j h_k dq^j dq^k \hat{b}_i.(no \;\; summation)$$

For a volume element it becomes

$$dV = \vec{b}_i \cdot \vec{b}_j \times \vec{b}_k = h_i h_j h_k \hat{b}_i \cdot \hat{b}_j \times \hat{b}_k dq^i dq^j dq^k = h_i h_j h_k dq^i dq^j dq^k.$$

Here we want to discuss the expressions for gradient, divergence and curl in the orthogonal curvilinear coordinates. The gradient is most easy. From

$$df = \nabla f \cdot d\vec{r} = \nabla f \cdot \vec{b}_i dq^i = (\nabla f)_i h_i dq^i \hat{b}_i = \frac{\partial f}{\partial q^i} dq^i.$$

One has

$$\nabla f = \frac{1}{h_i} \frac{\partial f}{\partial q^i} \hat{b}_i.$$

From Gauss theorem: $\nabla \cdot \vec{F}$=$\lim_{V\to 0} \frac{\oint_S \vec{F} \cdot \hat{n} dA}{V}$ Assume the eight corners the volume is A=(q^1, q^2, q^3), B=(q^1+dq^1, q^2, q^3), C=(q^1, q^2+dq^2, q^3), D=(q^1, q^2, q^3+dq^3), E=$(q^1+dq^1, q^2+dq^2, q^3)$, F=$(q^1, q^2+dq^2, q^3+dq^3)$, G=$(q^1+dq^1, q^2, q^3+dq^3)$, H=$(q^1+dq^1, q^2+dq^2, q^3+dq^3)$.

$$\begin{aligned}
\vec{F} \cdot dA(\square EBGH) &= \vec{F}_1(q^1+dq^1, q^2, q^3) h_2(q^1+dq^1, q^2, q^3) h_3(q^1+dq^1, q^2, q^3) dq^2 dq^3, \\
\vec{F} \cdot dA(\square ACFD) &= -\vec{F}_1(q^1, q^2, q^3) h_2(q^1, q^2, q^3) h_3(q^1, q^2, q^3) dq^2 dq^3, \\
\vec{F} \cdot dA(\square FDGH) &= \vec{F}_3(q^1, q^2, q^3+dq^3) h_1(q^1, q^2, q^3+dq^3) h_2(q^1, q^2, q^3+dq^3) dq^1 dq^2, \\
\vec{F} \cdot dA(\square ACEB) &= -\vec{F}_1(q^1, q^2, q^3) h_1(q^1, q^2, q^3) h^2(q^1, q^2, q^3) dq^1 dq^2, \\
\vec{F} \cdot dA(\square CGHF) &= \vec{F}_2(q^1, q^2+dq^2, q^3) h_1(q^1, q^2+dq^2, q^3) h_3(q^1, q^2+dq^2, q^3) dq^1 dq^3, \\
\vec{F} \cdot dA(\square ABGD) &= -\vec{F}_1(q^1, q^2, q^3) h_1(q^1, q^2, q^3) h_3(q^1, q^2, q^3) dq^1 dq^3,
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{V\to 0}\oint_S \vec{F}\cdot\hat{n}dA &= \frac{1}{h_1h_2h_3dq^1dq^2dq^3}[(F_1h_2h_3(q^1+dq^1,q^2,q^3)-F_1h_2h_3(q^1,q^2,q^3))dq^2dq^3\\
&+ (F_2h_1h_3(q^1,q^2+dq^2,q^3)-F_2h_1h_3(q^1,q^2,q^3))dq^1dq^3\\
&+ (F_3h_1h_2(q^1,q^2,q^3+dq^3)-F_3h_1h_2(q^1,q^2,q^3))dq^2dq^3]\\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial q^1}+\frac{\partial(h_1h_3F_2)}{\partial q^2}+\frac{\partial(h_1h_2F_3)}{\partial q^3}\right]
\end{aligned}$$

Now we will derive similar expression for curl. From Stock's theorem one has $\nabla\times\vec{F}=\lim_{A\to 0}\frac{\oint_C\vec{f}\cdot d\vec{l}}{A}$ Choose Four point $A=(q^1,q^2,q^3)$, $B=(q^1,q^2+dq^2,q^3)$, $C=(q^1+dq^1,q^2,q^3)$ and $D=q^1+dq^1,q^2+dq^2,q^3)$. Now we have C as combination of AC and CD and DB and BA.

$$\begin{aligned}
\int_A^C \vec{F}\cdot d\vec{l} &= F_1(q^1,q^2,q^3)h_1(q^1,q^2,q^3)dq^1,\\
\int_C^D \vec{F}\cdot d\vec{l} &= F_2(q^1+dq^1,q^2,q^3)h_2(q^1+dq^1,q^2,q^3)dq^2,\\
\int_D^B \vec{F}\cdot d\vec{l} &= -F_1(q^1,q^2+dq^2,q^3)h_1(q^1,q^2+dq^2,q^3)dq^1,\\
\int_B^A \vec{F}\cdot d\vec{l} &= -F_2(q^1,q^2,q^3)h_2(q^1,q^2,q^3)dq^2,
\end{aligned}$$

Hence

$$\oint_C \vec{F}\cdot d\vec{l} = (F_2h_2(q^1+dq^1,q^2,q^3)-F_2h_2(q^1,q^2,q^3))dq^2-(F_1h_1(q^1,q^2+dq^2,q^3)-F_1h_1(q^1,q^2,q^3))dq^1.$$

Hence

$$(\nabla\times\vec{F})_3 = \frac{1}{h_1h_2}\left(\frac{\partial(F_2h_2)}{\partial q^1}-\frac{\partial(F_1h_1)}{\partial q^2}\right).$$

Similar we have

$$\begin{aligned}
(\nabla\times\vec{F})_1 &= \frac{1}{h_2h_3}\left(\frac{\partial(F_3h_3)}{\partial q^2}-\frac{\partial(F_2h_2)}{\partial q^3}\right).\\
(\nabla\times\vec{F})_2 &= \frac{1}{h_1h_3}\left(\frac{\partial(F_1h_1)}{\partial q^3}-\frac{\partial(F_3h_3)}{\partial q^1}\right).
\end{aligned}$$

In short we can write down the following expression:

$$\nabla\times\vec{F} = \epsilon_{ijk}\frac{h_i\hat{b}_i}{h_1h_2h_3}\left(\frac{\partial F_kh_k}{\partial q^j}-\frac{\partial F_jh_j}{\partial q^k}\right).$$

Last, we need the expression for Laplacian. This is easy:

$$\begin{aligned}\nabla\cdot\nabla f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}(h_2h_3\nabla f_1)+\frac{\partial}{\partial q^2}(h_1h_3\nabla f_2)+\frac{\partial}{\partial q^3}(h_1h_2\nabla f_3)\right]\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q^1}\right)+\frac{\partial}{\partial q^2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial q^2}\right)+\frac{\partial}{\partial q^3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q^3}\right)\right]\end{aligned}$$

Example 1.9 Spherical coordinate: $r=\sqrt{x^2+y^2+z^2}$, $\theta=\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, $\phi=\tan^{-1}\left(\frac{y}{x}\right)$. Express gradient, curl, divergence and Laplaian in this coordinate.
Solution:

$$\vec{r}=r\sin\theta\cos\phi\hat{e}_1+r\sin\theta\sin\phi\hat{e}_2+r\cos\theta\hat{e}_3.$$

Hence

$$\begin{aligned}\vec{b}_r &= \sin\theta\cos\phi\hat{e}_1+\sin\theta\sin\phi\hat{e}_2+\cos\theta\hat{e}_3,\\ \vec{b}_\theta &= r\cos\theta\cos\phi\hat{e}_1+r\cos\theta\sin\phi\hat{e}_2-r\sin\theta\hat{e}_3,\\ \vec{b}_\phi &= -r\sin\theta\sin\phi\hat{e}_1+r\sin\theta\cos\phi\hat{e}_2.\end{aligned}$$

One can obtain that $h_r=1$, $h_\theta=r$ and $h_\phi=r\sin\theta$. Consequently one has

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial r}\hat{b}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}.\\ \nabla\cdot\vec{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial r^2\sin\theta F_r}{\partial r}+\frac{\partial r\sin\theta F_\theta}{\partial\theta}+\frac{\partial rF_\phi}{\partial\phi}\right]\\ &= \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.\\ \nabla\times\vec{F} &= \frac{1}{r\sin\theta}\left(\frac{\partial\sin\theta F_\phi}{\partial\theta}-\frac{\partial F_\theta}{\partial\phi}\right)\hat{b}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{\partial rF_\phi}{\partial r}\right)\hat{\theta}+\frac{1}{r}\left(\frac{\partial rF_\theta}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\hat{e}_\phi.\\ \nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.\end{aligned}$$

Exercise 1.4: Cylindrical coordinate: $\rho=\sqrt{x_1^2+x_2^2}$, $\phi=\tan^{-1}\left(\frac{x_2}{x_1}\right)$. $x_3=z$ Express gradient, curl, divergence and Laplaian in this coordinate.

Home Work 1.7: Parabolic cylindrical coordinate: $x_1=\sigma\tau$ and $x_2=\frac{\tau^2-\sigma^2}{2}$,$x_3=z$. Express gradient, curl, divergence and Laplacian in this coordinate.

Home Work 1.8: Elliptic cylindrical coordinate: $x_1{=}a\cosh\mu\cos\nu$ and $x_2{=}a\sinh\mu\sin\nu$, $x_3{=}z$ Express gradient, curl, divergence and Laplacian in this coordinate.

IX. SEPARATION OF VARIABLES OF LAPLACE OPERATOR

Here we want to show how to obtain three ordinary second order differential equations from the separation of variables of Laplace equation at different curvilinear coordinates. First let us do it at spherical coordinate. Here Φ is a function and its Laplacian at spherical coordinate is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

Now we assume Φ can be written as the follows: $\Phi{=}R(r)P(\theta)W(\phi)$. This is bold assumption, nevertheless let us try! We first obtain

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{Pr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{Wr^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2} = 0.$$

Multiplying $r^2\sin^2\theta$ and it becomes

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} = 0.$$

Observe the above equation we find that the first terms are independent on ϕ but the last term is only dependent on ϕ. Therefore we have

$$\frac{1}{W}\frac{d^2W}{d\phi^2} = -m^2.$$

The reason to choose the negative number is because otherwise one won't obtain periodic function of ϕ. Sequently we obtain

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) - m^2 = 0.$$

Now we divide the equation by $\sin^2\theta$:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = 0.$$

The first term is independent on θ but the other two terms are only dependent on θ, so that

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k.$$

Here k is arbitrary number. At the end of this process we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}P+kP=0.$$

To make our equation more elegant we choose the variable $\mu{=}\cos\theta$, $\frac{d}{d\theta}{=}{-}(1-\mu^2)^{1/2}\frac{d}{d\mu}$.

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\\=\;&\frac{1}{\sqrt{1-\mu^2}}\left[-(1-\mu^2)^{1/2}\frac{d}{d\mu}\right]\left[-(1-\mu^2)\frac{dP}{d\mu}\right]\\=\;&\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right).\end{aligned}$$

The solution of this equation is Legendre functions. Next we try to do similar thing at cylindrical coordinate. The Laplacian at cylindrical coordinate is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0.$$

Assume the separation of variable works: $\Phi=R(\rho)W(\phi)Z(z)$ Then we have

$$\frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{W\rho^2}\frac{\partial^2W}{\partial\phi^2}+\frac{1}{Z}\frac{\partial^2Z}{\partial z^2}=0.$$

The last term is only dependent on z and other terms are independent of z so that $\frac{1}{Z}\frac{\partial^2Z}{\partial z^2}=k^2$. The reason to choose positive number is because $z(z)$ must vanish when $z\to\pm\infty$. So that

$$\frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{W\rho^2}\frac{\partial^2W}{\partial\phi^2}+k^2=0.$$

Multiplying ρ^2 one obtains

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+k^2\rho^2+\frac{1}{W}\frac{\partial^2W}{\partial\phi^2}=0.$$

Now the first two terms are independent of ϕ and the last one only depends on ϕ so $\frac{1}{W}\frac{\partial^2W}{\partial\phi^2}=-m^2$. The reason to choose negative is because $W(\phi)$ must be periodic function. Now we have

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+k^2\rho R-\frac{m^2}{\rho}R=0.$$

To make the equation more elegant we choose the new variable: $x{=}k\rho$. The equation becomes

$$\begin{aligned}&k\rho\frac{\partial}{\partial(k\rho)}\left(k\rho\frac{\partial R}{\partial(k\rho)}\right)+k^2\rho R-\frac{km^2}{k\rho}R=0.\\&kx\frac{\partial}{\partial x}\left(x\frac{\partial R}{\partial x}\right)+kxR-\frac{km^2}{x}R=0.\end{aligned}$$

Namely we have

$$x\frac{d}{dx}\left(x\frac{dR}{dx}\right)+xR-\frac{m^2}{x}R=0.$$

The solution of this equation is called Bessel function.

Both of Legendre functions and Bessel functions are both well studied in this summer course. Next lecture we will first study the general theory of ordinary differential equation.

Home Work 1.9: Please operate the same procedure of separation of variable for the Laplace equation at elliptic cylindrical coordinate.

X. APPENDIX: DIVERGENCE IN CURVILINEAR COORDINATES: GENERAL CASE

The next task is how to write down the gradient, curl and divergence in term of arbitrary coordinate. The crucial pint is the basis vectors here are dependent on the position. Assume

$$\frac{\partial \vec{b}_i}{\partial q^j}=\frac{\partial x^m}{\partial q^i\partial q^j}\hat{e}_m=\Gamma^k_{ij}\vec{b}_k.$$

Note that

$$\frac{\partial \vec{b}_j}{\partial q^i}=\frac{\partial x^m}{\partial q^j\partial q^i}=\Gamma^k_{ji}\vec{b}_k.$$

It is easy to see Γ^k_{ij}=Γ^k_{ji}.

Then the differential of a vector in curvilinear coordinate becomes

$$\begin{aligned}\frac{\partial \overrightarrow{F}}{\partial q^i}&=\frac{\partial}{\partial q^i}\left(\tilde{F}^j\vec{b}_j\right)=\frac{\partial \tilde{F}^j}{\partial q^i}\vec{b}_j+\tilde{F}^j\frac{\partial \vec{b}_j}{\partial q^i}\\&=\frac{\partial \tilde{F}^j}{\partial q^i}\vec{b}_j+\tilde{F}^j\Gamma^k_{ji}\vec{b}_k=\left(\frac{\partial \tilde{F}^k}{\partial q^i}+\tilde{F}^j\Gamma^k_{ji}\right)\vec{b}_k.\end{aligned}$$

Therefore

$$\begin{aligned}\nabla\cdot\overrightarrow{F}&=\frac{\partial F_i}{\partial x^i}=\frac{\partial \overrightarrow{F}}{\partial x^i}\cdot\hat{e}_i=\frac{\partial \overrightarrow{F}}{\partial q^k}\cdot\hat{e}_i\frac{\partial q^k}{\partial x^i}\\&=\left(\frac{\partial \tilde{F}^k}{\partial q^i}+\tilde{F}^j\Gamma^k_{ji}\right)\vec{b}_k\cdot\hat{e}_i\frac{\partial q^k}{\partial x^i}=\left(\frac{\partial \tilde{F}^k}{\partial q^i}+\tilde{F}^j\Gamma^k_{ji}\right)\frac{\partial x^i}{\partial q^k}\frac{\partial q^k}{\partial x^i}\\&=\frac{\partial \tilde{F}^i}{\partial q^i}+\tilde{F}^j\Gamma^i_{ji}.\end{aligned}$$

So we need evaluate Γ^k_{ij}. This is a little bit tricky. Remember that $\tilde{g}_{ij} = \vec{b}_i \cdot \vec{b}_j$. Hence

$$\frac{\partial \tilde{g}_{ij}}{\partial q^k} = \frac{\partial \vec{b}_i}{\partial q^k} \cdot \vec{b}_j + \frac{\partial \vec{b}_j}{\partial q^k} \cdot \vec{b}_i = \Gamma^m_{ik} \vec{b}_m \cdot \vec{b}_j + \Gamma^m_{jk} \vec{b}_m \cdot \vec{b}_i = \Gamma^m_{ik} \tilde{g}_{mj} + \Gamma^m_{jk} \tilde{g}_{mi}.$$

Similarly we have

$$\frac{\partial \tilde{g}_{kj}}{\partial q^i} = \Gamma^m_{ki} \tilde{g}_{mj} + \Gamma^m_{ji} \tilde{g}_{mk}. \quad \frac{\partial \tilde{g}_{ik}}{\partial q^j} = \Gamma^m_{ij} \tilde{g}_{mk} + \Gamma^m_{kj} \tilde{g}_{mi}.$$

One has

$$\frac{\partial \tilde{g}_{ij}}{\partial q^k} + \frac{\partial \tilde{g}_{kj}}{\partial q^i} - \frac{\partial \tilde{g}_{ik}}{\partial q^j} = 2\Gamma^m_{ki} \tilde{g}_{jm}.$$

So one has

$$\frac{1}{2} \tilde{g}^{nj} \left(\frac{\partial \tilde{g}_{ij}}{\partial q^k} + \frac{\partial \tilde{g}_{kj}}{\partial q^i} - \frac{\partial \tilde{g}_{ik}}{\partial q^j} \right) = \frac{1}{2} \tilde{g}^{nj} \left(2\Gamma^m_{ki} \tilde{g}_{jm} \right) = \Gamma^n_{ki}.$$

Therefore

$$\begin{aligned} \Gamma^i_{ij} &= \frac{1}{2} \tilde{g}^{il} \left(\frac{\partial \tilde{g}_{jl}}{\partial q^i} + \frac{\partial \tilde{g}_{il}}{\partial q^j} - \frac{\partial \tilde{g}_{ji}}{\partial q^l} \right) \\ &= \frac{1}{2} \tilde{g}^{il} \left(\frac{\partial \tilde{g}_{jl}}{\partial q^i} + \frac{\partial \tilde{g}_{il}}{\partial q^j} \right) - \frac{1}{2} \tilde{g}^{li} \left(\frac{\partial \tilde{g}_{jl}}{\partial q^i} \right) = \frac{1}{2} \tilde{g}^{il} \frac{\partial \tilde{g}_{il}}{\partial q^j} \end{aligned}$$

The determinant of $\tilde{g}_{ij}$ is expressed as

$$\tilde{g} = \det(\tilde{g}_{ij}) = \sum_i \tilde{g}_{ij} A^{ij}, \quad \frac{\partial \tilde{g}}{\partial \tilde{g}_{ij}} = A^{ij}.$$

Here A^{ij} is the cofactor. Remember $\tilde{g}^{ij}$ is the inverse of $\tilde{g}_{ij}$. Therefore

$$\tilde{g}^{ij} = \frac{A^{ij}}{\det g_{ij}}.$$

Hence one has

$$\frac{1}{\tilde{g}} \frac{\partial \tilde{g}}{\partial g_{ij}} = \tilde{g}^{ij}.$$

So we have

$$\frac{1}{2} \tilde{g}^{il} \frac{\partial \tilde{g}_{il}}{\partial q^j} = \frac{1}{2\tilde{g}} \frac{\partial \tilde{g}}{\partial q^j}.$$

Therefore

$$\nabla \cdot \overrightarrow{F} = \frac{\partial \tilde{F}^i}{\partial q^i} + \tilde{F}^j \Gamma^i_{ji} = \frac{\partial \tilde{F}^i}{\partial q^i} + \frac{1}{2\tilde{g}} \frac{\partial \tilde{g}}{\partial q^i} \tilde{F}^j = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial q^i} \left(F^{(i)} \sqrt{\tilde{g}} \right).$$