Lecture II: Ordinary Differential Equation

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This lecture introduce the basic elements of ordinary differential equation.

I. HOMOGENEOUS SECOND ORDER EQUATION

In physics the main object one has to deal with is essentially differential equations. Usually they are the second order differential equations. Here we first study homogeneous case. The general form of homogeneous second order ordinary differential equation reads as

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0.$$

For a second order ordinary differential equation one needs two initial conditions to specify the solution. In other words, there should be two independent solutions $y_1(x)$ and $y_2(x)$. In general $c_1y_1(x) + c_2y_2(x)$ is the general form of the solution and two initial conditions will determine the values of c_1 and c_2.

Example 2.1: Please find the general solution of $\frac{d^2y}{dx^2} - 2b\frac{dy}{dx} + ky = 0$.
Solution:
Assume $y = e^{izx}$ then

$$(-z^2 - 2ibz + k)e^{izx} = 0 \Longrightarrow z = ib \pm (k - b^2)^{1/2}.$$

Such that $y{=}e^{-bx}e^{\sqrt{k-b^2}x}$ and $y{=}e^{-bx}e^{-\sqrt{k-b^2}x}$. Therefore the solution is

$$y(x) = c_1e^{-bx}e^{i\sqrt{k-b^2}x} + c_2e^{-bx}e^{-i\sqrt{k-b^2}x}$$

The value of c_1 and c_2 is decided by two initial conditions. When one solution is known, how to obtain another independent one? For example, in Example 2.1 when $k = b^2$, the two solutions become one, one needs to know the another independent solution. Here is the standard way:

Wornskian:

The Wornskian is defined as

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'.$$

Interesting it has the following property.

$$\begin{aligned}
\frac{1}{W(x)}\frac{dW(x)}{dx} &= \frac{1}{W(x)}(y_1 y_2'' - y_2 y_1'') \\
&= \frac{1}{W(x)}(y_1[-f(x)y_2' - g(x)y_2] - y_2[-f(x)y_1' - g(x)y_1]) = \frac{f(x)W(x)}{W(x)} = f(x), \\
W(x) &= W(x_0)\exp\left(-\int_{x_0}^{x} f(\xi)d\xi\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{d}{dx}\left(\frac{y_2}{y_1}\right) &= \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W}{y_1^2}. \\
y_2(x) &= y_1(x)\int^{x}\frac{W(\xi)}{y_1^2(\xi)}d\xi.
\end{aligned}$$

The following example is typical case.

Example 2.2: $y'' - 4y' + 4y = 0$ has one solution e^{2x}. please find another solution.
Solution:
$f(x)$=4.

$$\begin{aligned}
W(x) &= W(0)\exp\left(\int_0^x d\xi 4 d\xi\right) = W_0 e^{4x}. \\
y_2(x) &= e^{2x}\int^x \frac{W_0 e^{4\xi}}{(e^{2\xi})^2}d\xi = e^{2x}W_0\int^x d\xi = W_0 x e^{2x}.
\end{aligned}$$

Another method is to assume $y_2(x)$=$u(x)y_1(x)$ and sometime it is relatively easy to obtain $u(x)$ instead of y_2. Since $y_2'' + f y_2' + g y_2 = 0 \Longrightarrow u(y_1'' + f y_1' + g y_1) + u'' y_1 + u'(2y_1' + f y_1) = 0$. Since y_1 is also solution therefore we have

$$u'' y_1 + u'(2y_1' + f y_1) = 0.$$

That is

$$\frac{u''(\xi)}{u'(\xi)} = -2\frac{y_1'(\xi)}{y_1(\xi)} - f(\xi).$$

Integrate over ξ to eta we have

$$\ln u'(\eta) = -2\ln y_1(\eta) - \int^{\eta} f(\xi)d\xi.$$

That is $u'(\eta) = \frac{1}{y_1^2(\eta)} e^{-\int^{\eta} f(\xi)d\xi}$. Next we integrate over η to x and obtain,

$$u(x) = \int^{x} \frac{1}{y_1^2(\eta)} e^{-\int^{\eta} f(\xi)d\xi} d\eta.$$

We reach the exact same expression. However this method can be extended to nonhomogeneous situations.

Example 2.3: $(1-x^2)y'' - 2xy' + 2y = 0$ has one solution x. please find another solution.
Solution:
$f(x)=\frac{-2x}{1-x^2}$ and $g(x)=\frac{2}{1-x^2}$. So

$$u(x) = \int^{x} \frac{1}{\eta^2} \exp^{-\int^{\eta} \frac{-2\xi}{1-\xi^2} d\xi} d\eta$$

$$\int^{\eta} \frac{2\xi}{1-\xi^2} d\xi = \int^{\eta} \left[\frac{1}{1-\xi} - \frac{1}{1+\xi}\right] d\xi = \ln(1-\eta^2).$$

$$\begin{aligned} u(x) &= \int^{x} \frac{1}{\eta^2} \exp^{-\ln(1-\eta^2)} d\eta = \int^{x} \frac{1}{\eta^2} \frac{1}{1-\eta^2} d\eta = \int^{x} \left(\frac{1}{\eta^2} \frac{1}{1-\eta^2}\right) d\eta \\ &= \int^{x} \left(\frac{1}{\eta^2} + \frac{1}{1-\eta^2}\right) d\eta = \int^{x} \left(\frac{1}{\eta^2} + \frac{1}{2}\left(\frac{1}{1-\eta} + \frac{1}{1+\eta}\right)\right) d\eta \\ &= -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

$$y_2(x) = u(x)y_1(x) = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Exercise 2.1: Please find the general solution of $x^2y'' - 2xy' + 2y = 0$.

Home Work 2.1: Please find the general solution of $(1-x^2)y'' - xy' + y = 0$.

Home Work 2.2: Please find the general solution of $xy'' - (2x+1)y' + (x+1)y = 0$.

II. POWER SERIES SOLUTION

When $x = x_0$ which is not singularity of $f(x)$ and $g(x)$ one can expand the solution as the series of $(x-x_0)^n$ and derive the relations of the coefficients.

Example 2.4: Solve the Hermite equation: $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y = 0$.
Solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n. \\ y' &= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n. \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \end{aligned}$$

Insert them into the equation one has

$$(2a_2 + 2\alpha a_0)x^0 + \sum_{n=1}^{\infty}[(n+1)(n+2)a_{n+2} - 2na_n + 2\alpha a_n]x^n = 0.$$

Therefore one has

$$a_2 = -\alpha a_0. \;\; a_{n+2} = 2a_n\frac{n-\alpha}{(n+1)(n+2)}.$$

Hence for even $n = 2k$:

$$a_{2k} = (-2)^k\frac{\alpha(\alpha-2)....(\alpha-2k+2)}{2k!}a_0.$$

For odd $n = 2k+1$:

$$a_{2k+1} = (-2)^k\frac{\alpha(\alpha-2)....(\alpha-2k+1)}{(2k+1)!}a_1.$$

Therefore we find two independent solutions. If $\alpha = 2p$, p is an integer then $a_{2p+2} = 0$. The solution with even term becomes polynomial. It is called $H_{2p}(x)$ Hermite polynomial.

Homework 2.3: Please find the power series solution of Airy function: $y'' + xy = 0$.
Homework 2.4: Please find the power series solution of Airy function:
$(1+x^2)y'' + 2xy' - 2y = 0$.
Method of Frobenius: When some singular points of $f(x)$ and/or $g(x)$ occur then the ordinary power series method fails. When $(x-x_0)f(x)$ and $(x-x_0)^2g(x)$ are analytic functions then x_0 is called regular singularity. In this case one may choose alternative one. namely one choose

$$y(x) = (x-x_0)^p\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Here p can be positive or negative, integer or non- integer even complex number is allowed. This is called "method of Frobenius".

Example 2.5:Applying method of Frobenius to obtain the solution valid for $x > 1$ for the following equation:

$$(x^2 - x)\frac{d^2y}{dx^2} + \left(2x - \frac{1}{2}\right)\frac{dy}{dx} + \frac{1}{4}y = 0$$

Solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+p}. \\ y' &= \sum_{n=0}^{\infty} (n+p) a_n x^{n+p-1}. \\ y'' &= \sum_{n=0}^{\infty} (n+p-1)(n+p) a_n x^{n+p-2}. \end{aligned}$$

Insert them one has

$$\sum_{n=0}^{\infty} (n+p)(n+p-1)a_n x^{n+p} - (n+p)(n+p-1)a_n x^{n+p-1} + 2(n+p)a_n x^{n+p}$$
$$-\frac{1}{2}(n+p)a_n x^{n+p-1} + \frac{1}{4}a_n x^{n+p} = 0.$$

$$\left(-p(p-1) + \frac{1}{2}p\right)x^{p-1} + \sum_{n=0}^{\infty}[(n+p)(n+p-1)a_n - (n+p+1)(n+p)a_{n+1}$$
$$+\ 2(n+p)a_n - \frac{1}{2}(n+p+1)a_{n+1} + \frac{1}{4}a_n]x^{n+p} = 0$$

.

The coefficient of the first term must be zero. It is called **indicial equation**. From it we know $p = 0$ or $p = \frac{1}{2}$. As $p = 0$

$$a_{n+1} = \frac{2n+1}{2(n+1)}a_n = \frac{(2n+1)!!}{2^{n+1}(n+1)!}a_0.$$

For $p = 1/2$

$$a_{n+1} = \frac{n+1}{n+3/2}a_n = \frac{2^{n+1}(n+1)!}{(2n+3)!!}a_0.$$

Hence there are two independent solutions

$$\begin{aligned} y_1 &= a_0 \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} x^n. \\ y_2 &= \sqrt{x} a_0 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!!} x^n. \end{aligned}$$

Home Work 2.5: Please find the two solutions of $2x^2y'' + x(2x+1)y' - y = 0$ by method of Frobenius.

When the roots of indicial equation has repeated roots or two root differing by an integer then the method of Frobenius fails to give two independent solutions. In this case, the second solution owns this form:

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} a_n x^{n+p}.$$

The reason of choosing such a ansatz is because $\ln x$ approaches infinity slower than any x^{-p}, $p > 0$. It can be easily realized by the fact $\lim_{x\to 0} \frac{\ln x}{x^{-p}} = \lim_{x\to 0} \frac{1/x}{(-p)x^{-p-1}} \to 0$.

Example 2.6:Applying method of Frobenius to obtain the solution valid for $x > 1$ for the following equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0$$

Solution:

Change the variable $w = x - 1$. The equation becomes

$$(w+2)w\frac{d^2y}{dw^2} + 2(w+1)\frac{dy}{dw} = 0.$$

Assume $y = \sum_{n=0}^{\infty} a_n w^n$.

$$\sum_{n=0}^{\infty}(n+p)(n+p-1)a_n w^{n+p} + 2(n+p)(n+p-1)a_n w^{n+p-1} + 2(n+p)a_n w^{n+p} + 2(n+p)a_n w^{n+p-1} = 0$$

The lowest term is w^{p-1}:

$$\begin{aligned} & [2p(p-1)+2p]a_0 w^{p-1} \\ + & \sum_{n=0}^{\infty}[(n+p)(n+p-1)a_n + 2(n+p+1)(n+p)a_{n+1} + 2(n+p)a_n + 2(n+p+1)a_{n+1}]w^{n+p} = 0. \end{aligned}$$

The solution is $p = 0$. And $a_{n+1} = a_n \frac{-n}{2(n+1)}$. However this solution $\sum_{n=0}^{\infty} a_n w^n = a_0$ is a trivial solution. The second solution now is assumed as

$$y_2 = y_1 \ln w + w^p \sum_{n=0}^{\infty} a_n w^n.$$

Take $y_1 = 1$ Insert to equation one has

$$\left(-\frac{w+2}{w} + 2\frac{w+1}{w}\right) + \sum_{n=0}^{\infty}[(n+p)(n+p+1)a_n + 2(n+p+1)^2 a_{n+1}]w^{n+p} = 0.$$

It becomes

$$1 + 2a_1 + \sum_{n=1}^{\infty}[n(n+1)a_n + 2(n+1)^2 a_{n+1}]w^n = 0$$

So $n(n+1)a_n = -2(n+1)^2 a_{n+1}$ which means $a_1 = -\frac{1}{2}$ besides we have

$$a_n = -\frac{n-1}{2n}a_{n-1} = (-1)^{n-1}\frac{1}{n2^{n-1}}a_1 = \frac{(-1)^n}{2^n n}.$$

Therefore the second solution is

$$\begin{aligned} y_2(w) &= \ln w + \sum_{n=1}^{\infty}(-1)^n \left(\frac{w}{2}\right)^n \frac{1}{n} \\ &= \ln w + \ln\left(1 + \frac{w}{2}\right) = \ln\left(\frac{x-1}{x+1}\right). \end{aligned}$$

Home Work 2.6: Please find the two solutions of $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$.

Asymptotic method:

What if we want to have solution valid for large value x? We can change variable $u = 1/x$. A solution valid in a neighborhood of $u = 0$ is a solution valid about the "point at infinity".

Example 2.7: To obtain the solution at the point at infinity of the following equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{m^2}{x^2}\right) y = 0$$

. Solution:

For large x the terms with $1/x$ and $1/x^2$ become very small. The equation simplifies to

$$\frac{d^2 y_\infty}{dx^2} - y_\infty = 0.$$

It is trivial to see $y_\infty = e^{\pm x}$. now let us define $y = v(x)e^{\pm x}$ The equation becomes

$$v'' + \left(\frac{1}{x} \pm 2\right) v' + \frac{1}{x}\left(\pm - \frac{m^2}{x}\right) v = 0.$$

Now change variable to $u = 1/x$. Then

$$\begin{aligned} \frac{dv}{dx} &= \frac{dv}{du}\frac{du}{dx} = -u^2\frac{dv}{du}, \\ \frac{d^2v}{dx^2} &= \frac{d}{du}\left(-u^2\frac{dv}{du}\right)\frac{du}{dx} = u^4\frac{d^2v}{du^2} + 2u^3\frac{dv}{du}. \end{aligned}$$

Hence the equation becomes

$$u^3\frac{d^2v}{du^2} + (u^2 \mp 2u)\frac{dv}{du} + (\pm 1 - m^2 u)v = 0.$$

Insert $v(u)=\sum_{n=0}^{\infty} a_n u^{n+p}$. We have

$$\sum_{n=0}^{\infty}[(n+p)(n+p-1)a_n u^{n+p+1} + (n+p)a_n u^{n+p+1} \mp 2(n+p)a_n u^{n+p} \pm a_n u^{n+p} - m^2 a_n u^{n+p+1} = 0.$$

The lowest term is u^p. Its coefficient is $(\mp 2p \pm 1)a_0$. So $p = 1/2$. Consequently we have

$$a_{k+1} = \mp\frac{4m^2 - (2k+1)^2}{8(k+1)} a_k.$$

They are the two solutions.

Home Work 2.7: Please discuss the nature of the point $x=\infty$ for $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$.

III. STURM-LIOUVILLE THEORY

In many physical systems the following forms of differential equations emerge.

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}\right) - g(x)y + \lambda w(x)y = 0.$$

Here $w(x) \geq 0$ on the range $[a, b]$. The solution $y(x)$ satisfies the following boundary conditions:

$$a_1 y(x = a) + b_1 y'(x = a) = 0,\ a_2 y(x = b) + b_2 y'(x = b) = 0.$$

a_i=0 is called Neunmann conditions. b_i=0 is called Dirichlet conditions.

There is a set of function $y_n(x)$ which is the solution of the above equation with $\lambda = \lambda_n$. Those functions are called eigenfunctions and λ_n are called eigenvalues. For the eigenfunctions with different eigenvalues we can prove that they are orthogonal. Let us start from the following:

$$\begin{aligned}
&\frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) - g(x)y_m + \lambda_m w(x) y_m = 0. \\
&\frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) - g(x)y_n + \lambda_n w(x) y_n = 0. \\
&y_n\frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) + (\lambda_m - \lambda_n) w y_m y_n = 0. \\
&\int_a^b y_n\frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) dx = (\lambda_n - \lambda_m)\int_a^b w(x) y_m y_n dx
\end{aligned}$$

Remember that

$$\int_a^b y_n\frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) dx = y_n f(x)\frac{dy_m}{dx}|_a^b - \int_a^b f(x)\frac{dy_n}{dx}\frac{dy_m}{dx}dx$$

Therefore when $n \neq m$ and $\lambda_m \neq \lambda_n$ then

$$\int_a^b w(x) y_m(x) y_n(x) dx = 0.$$

Hence we have

$$\begin{aligned}
&\int_a^b y_n\frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) dx \\
= \ & y_n(b)f(b)\frac{dy_m(b)}{dx} - y_n(a)f(a)\frac{dy_m(a)}{dx} - \int_a^b f(x)\frac{dy_n}{dx}\frac{dy_m}{dx}dx \\
- \ & y_m(b)f(b)\frac{dy_n(b)}{dx} + y_m(a)f(a)\frac{dy_n(a)}{dx} + \int_a^b f(x)\frac{dy_m}{dx}\frac{dy_n}{dx}dx \\
= \ & y_n(b)f(b)\frac{-a_2}{b_2}y_m(b) - y_n(a)f(a)\frac{-a_1}{b_1}y_m(a) - y_m(b)f(b)\frac{-a_2}{b_2}y_n(b) + y_m(a)f(a)\frac{-a_1}{b_1}y_n(a) \\
= \ & 0
\end{aligned}$$

Any well-behaved function $f(x)$ defined in $[a, b]$ can be expanded in a series of eigenfunctions:

$$f(x) = \sum_{n=0}^{\infty} a_n y_n(x).$$

here

$$a_n = \frac{\int_a^b f(x) y_n(x) w(x) dx}{\int_a^b w(x) y_n(x) y_n(x) dx}.$$

Home Work 2.8 When $f(x)$, $g(x)$ and $w(x)$ are all real and $w(x) \geq 0$. Please prove that the eigenvalue λ_n is real.

IV. NONHOMOGENEOUS SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

From this section we start to study the non-homogeneous case:

$$\frac{d^2 y}{dx^2} + f(x)\frac{dy}{dx} + g(x) y = h(x).$$

In general if one find a solution of the above the equation $y_p(x)$ which is called "particular solution", the the combination of y_p and y_c which is the solution of $\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0$ is also a solution. It is natural because to determine the solution one needs two initial conditions and consequently we need two free parameters. Therefore

$$y(x) = y_p(x) + c_1 y_{c,1} + c_2 y_{c,2}.$$

We have discuss the method of acquiring y_c, so we will focus on the method of acquiring the particular solution here. To assume $y(x)$=$u_1(x)y_1(x)$ + $u_2(x)y_2(x)$ here $y_1(x)$ and y_2 are the two independent solutions of $\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0$. Furthermore we request : $u_1' y_1 + u_2' y_2 = 0$. Then we have

$$u_1(y_1'' + f y_1' + g y_1) + u_2(y_2'' + f y_2' + g y_2) + u_1 y_1' + u_2' y_2' = h(x).$$

It actually becomes

$$u_1' y_1' + u_2' y_2' = h(x), \;\; u_1' y_1 + u_2' y_2 = 0.$$

It causes

$$u_1' = \frac{-y_2 h(x)}{W(y_1, y_2)}, \;\; u_2' = \frac{-y_1 h(x)}{W(y_1, y_2)}.$$

Therefore

$$y(x) = y_1(x)\int^x \frac{-y_2(\xi)h(\xi)}{W(y_1(\xi), y_2(\xi))}d\xi + y_2(x)\int^x \frac{-y_1(\xi)h(\xi)}{W(y_1(\xi), y_2(\xi))}d\xi.$$

Home Work 2.9: Find the general solution of the following equation:

$$y'' + 2y = \csc x.$$

Home Work 2.10: Find the general solution of the following equation:

$$(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2.$$

V. GREEN FUNCTIONS

A powerful way to obtain the particular solution of the nonhomogeneous ordinary differential equation is to apply the Green functions. Green functions are defined as the solutions of the following equation:

$$\left(\frac{d^2}{dx^2} + f(x)\frac{d}{dx} + g(x)\right) G(x, x') = -4\pi\delta(x - x').$$

For different initial conditions there will be different forms of the Green functions. If the initial condition is specified then one can apply it because the following expression is automatically the solution we look for.

$$y(x) = \int G(x, x')h(x')dx'.$$

Example 2.8: Please apply Green function to solve $m\frac{dv}{dt} + \alpha v = F(t)$. $v = 0$ as t=$\pm\infty$.
Solution:
First we need to obtain the Green function $G(t, t')$ satisfying $m\frac{dG}{dt} + \alpha G = \delta(t - t')$. For $t \neq t'$ the delta function is zero and the equation is $m\frac{dG(t,t')}{dt} + \alpha G(t, t') = 0$. Which has the form $G(t, t')$=$A\exp\left(-\frac{\alpha}{m}t\right)$. This form approaches zero as $t \to +\infty$ but blows up as

$t \to -\infty$. Since $v = 0$ when $t \to -\infty$ so we have $G(t, t') = 0$ when $t < t'$. The next step is to integrate the differential equation from $t' - \epsilon$ to $t' + \epsilon$.

$$\int_{t'-\epsilon}^{t'+\epsilon} \left(m\frac{dG(t,t')}{dt} + \alpha G(t,t') \right) dt = mG|_{t'-\epsilon}^{t'+\epsilon} + \alpha \int_{t'-\epsilon}^{t'+\epsilon} Gdt = 1.$$

Remember $G(t' - \epsilon, t')$=0. Since $| \int_{t'-\epsilon}^{t'+\epsilon} G(t)dt| \leq max|G|(2\epsilon) \to 0$. So we have

$$m \left[A \exp\left(-\frac{\alpha}{m} t' \right) - 0 \right] = 1.$$

A=$\frac{1}{m} \exp\left(\frac{\alpha}{m} t'\right)$. At the end

$$\begin{aligned} G(t,t') &= 0, if t \leq t' \\ G(t,t') &= \frac{1}{m} \exp\left(-\frac{\alpha}{m}(t-t') \right), if t > t'. \end{aligned}$$

So that

$$v(t) = \int_{-\infty}^{\infty} G(t,t')F(t')dt' = \int_{\infty}^{t} G(t,t')F(t')dt'.$$

Home Work 2.11: If y(x=0)=y(x=L)=0. Please find the green function as the solution of $\frac{d^2y}{dx^2} + k^2 y = \delta(x - x')$.

Another way to obtain the Green function is to apply Strum-Liouville theory. Since the Green function should satisfy

$$\frac{d}{dx}\left(f(x)\frac{dG}{dx} \right) - g(x)G + \lambda w(x)G = -4\pi\delta(x - x').$$

Here we expand it with the eigenfunctions

$$G(x, x') = \sum_{n=0}^{n=\infty} \gamma_n(x')y_n(x).$$

Remember that

$$\frac{d}{dx}\left(f(x)\frac{dy_n}{dx} \right) - g(x)y_n + \lambda_n w(x)y_n = 0.$$

So from

$$\sum_{n=0}^{\infty} \gamma_n(x')\frac{d}{dx}\left(f(x)\frac{dy_n}{dx} \right) - g(x)y_n + \lambda w(x)y_n = -4\pi\delta(x - x').$$

we have

$$\gamma_n(x')[-\lambda_n w(x)y_n(x) + \lambda w(x)y_n(x)] = -4\pi\delta(x - x').$$

Note multiply $y_m(x)$ then integrate over x the above equation from a to b

$$\sum_{n=0}^{\infty}(\lambda-\lambda_n)\gamma_n(x')\int_a^b w(x)y_n(x)y_m(x)dx = -4\pi\int_a^b \delta(x-x')y_m(x)dx.$$
$$\sum_{n=0}^{\infty}(\lambda-\lambda_n)\gamma_n(x')\delta_{nm} = -4\pi y_m(x').$$

Therefore

$$\gamma_m(x') = \frac{4\pi y_m(x')}{\lambda_m-\lambda},$$
$$G(x,x') = \sum_{n=0}^{\infty}\frac{4\pi y_n(x')y_n(x)}{\lambda_n-\lambda}.$$

Example 2.9: If y(x=0)=y(x=L)=0. Please find the green function as the solution of $\frac{d^2y}{dx^2}+k^2y=\delta(x-x')$. Please use Sturm-Liouville theory.

Solution:

$$\frac{d^2y_n}{dx^2}+\lambda_n y=0.$$

To satisfy the boundary conditions we obtain

$$y_n(x)=c_n\sin\left(\frac{n\pi x}{L}\right).$$

To extract c_n we note that

$$\int_0^L y_n(x)y_n(x)dx = 1. \Longrightarrow c_n^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = 1.$$
$$c_n = \sqrt{\frac{2}{L}}.$$

Hence

$$G(x,x') = \frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{k^2-\left(\frac{n\pi}{L}\right)^2}.$$

Example 2.10: Poisson Equation : $\nabla^2\Phi{=}{-}\rho(x)/\epsilon_0$ with boundary condition $\Phi(\vec{x})=0$ at the boundary. Apply Delta function to solve it.

Solution:

Here we define $\nabla^2 G(\vec{x},\vec{x'})=-4\pi\delta(\vec{x}-\vec{x'})$. From

$$\int_V(\Phi(x)\nabla^2G(x,x')-G(x,x')\nabla^2\Phi(x))dV = \int_s(\Phi(x)\nabla G(x,x')-G(x,x')\nabla\Phi 9x))\cdot\hat{n}dS.$$

$$\int_V \left[\Phi(x)[-4\pi\delta^3(\vec{x}-\vec{x'}) - G(x,x')\left(\frac{-\rho(x)}{\epsilon_0}\right)\right] dV = \int_S (\Phi(x)\nabla G(x,x') - G(x,x')\nabla\Phi(x))\cdot\hat{n} dS.$$

$$-\Phi(\vec{x'}) + \frac{1}{4\pi\epsilon_0}\int_V G(\vec{x},\vec{x'})\rho(x)dV = \frac{1}{4\pi}\int_S (\Phi(x)\nabla G(x,x') - G(x,x')\nabla\Phi(x))\cdot\hat{n}dS.$$

$\Phi(\vec{x})$ vanishes on the boundary so we choose $G_D(\vec{x},\vec{x'}) = 0$ on the boundary and obtain

$$\Phi(\vec{x'}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\vec{x},\vec{x'})\rho(\vec{x})dV.$$

Home Work 2.12 If $\nabla\Phi\cdot\hat{n}$ is specified on the boundary, please solve Poisson equation.