Lecture III: Complex Analysis

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

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This lecture introduce the basic elements of complex analysis.

I. COMPLEX PLANE

A complex number can be viewed as a point or position vector in a two-dimensional Cartesian coordinate system called the complex plane or Argand diagram, named after Jean-Robert Argand. The numbers are conventionally plotted using the real part as the horizontal component, and imaginary part as vertical. Namely we have

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

here we apply this relation $e^{i\theta} = \cos\theta + i\sin\theta$ derived from Taylor expansion. Naturally we have the following relations:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Remember the definition of the hyperbolic functions:

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2},\ \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$$

Thus we have

$$\cos(i\theta) = \cosh\theta,\ \sin(i\theta) = i\sinh\theta,\ \cosh(i\theta) = \cos\theta,\ \sinh(i\theta) = i\sin\theta.$$

Since $e^{i2\pi} = 1$ therefore one finds that $z = re^{\imath\theta} = re^{i(\theta+2n\pi)}$. with integer n. If we looks for the $m-th$ root of this number:

$$z^{1/n} = r^{1/n}e^{i\theta/m}e^{i2n\pi/m}.$$

So there are m distinct roots of any nonzero number. The absolute value of a complex number: $|z| = \sqrt{x^2+y^2} = r$ is the length of the corresponding vector. Consequently we have the following identities:

$$|z_1 + z_2| \leq |z_1| + |z_2|,\ |z_1 - z_2| \geq ||z_1| - |z_2||.$$

The complex conjugate of a complex number is $z^* = x - iy = re^{-i\theta}$. The inverse of complex number is $1/z = \frac{x-iy}{x^2+y^2} = \frac{1}{r}\cdot e^{-i\theta}$. Naturally we have $z^*z = r^2 \geq 0$.

Example 3.1: Please show that $\cos n\theta = \sum_{k=0}^{n/2} \frac{n!}{2k!(n-2k)!}(-1)^k \cos^{n-2k}\theta \sin^{2k}\theta$. Here n is even.

Solution:

$$\begin{aligned}\left(e^{i\theta}\right)^n &= e^{in\theta} = \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(i)^j \sin^j\theta \cos^{n-j}\theta \\ &= \sum_{k=0}^{n/2} \frac{n!}{2k!(n-2k)!}(-1)^k \sin^{2k}\theta \cos^{n-2k}\theta \\ &+ i \sum_{k=0}^{(n/2)-1} \frac{n!}{(2k+1)!(n-2k-1)!}(-1)^k \sin^{2k+1}\theta \cos^{n-2k-1}\theta\end{aligned}$$

So that

$$\cos n\theta = \sum_{k=0}^{n/2} \frac{n!}{2k!(n-2k)!}(-1)^k \cos^{n-2k}\theta \sin^{2k}\theta.$$

Home Work 3.1: Please show that

(a) $\sin^{-1} z = -i\ln[iz + \sqrt{1-z^2}]$. (b)$\cosh^{-1} z = \ln[z + \sqrt{z^2-1}]$. (c) $\tanh^{-1} z = \frac{1}{2}\ln\left[\frac{1+z}{1-z}\right]$.

II. COMPLEX FUNCTIONS AS MAPPINGS

A complex function $f(z)$ takes the number $z = x + iy$ and generates another complex number $w = u + iv$. This is a mechanism to setup a a correspondence between two points in the complex plane, therefore it is naturally to think a complex function as a mapping which maps complex plane to itself.

Example 3.2: Describe the mappings (a) $w = \frac{1}{z}$. (b) $w = \ln z$.

Solution:

(a) $w = 1/z$. $z = re^{i\theta}$, $w = \frac{1}{r}e^{-i\theta}$. Another way to see is $w = u + iv$. $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$.

(b) $w = \ln z = \ln r + i\theta$. or $w = u + iv$, $u = \frac{1}{2}\ln(x^2+y^2)$, $v = \tan^{-1}\frac{y}{x}$.

A complex function is continuous at a when any z satisfying $|z-a| \leq \delta$ such that $|f(z)-f(a)| \leq \epsilon$ for any positive ϵ. Roughly speaking the nearby point around of $z=a$ is mapped to the neighborhood of $f(a)$. A simply way to prove it is to prove $|f(z)-f(a)| \to 0$ as $|z-a| \leq \rho \to 0$.

Example 3.2 Prove that $f(z)=1/(z+c)$ is continuous except at z=-c.
Solution:
If $|z-a| \leq \rho$. Then

$$\begin{aligned} |f(z)-f(a)| &= |\frac{1}{z+c} - \frac{1}{a+c}| = |\frac{a-z}{(z+c)(a+c)}| = \frac{|a-z|}{|z+c||a+c|} \leq \frac{\rho}{|a+c||(z-a)+(a+c)|} \\ &\leq \frac{\rho}{|a+c|||a+c|-\rho|}. \end{aligned}$$

Therefore as long as $|a+c| \neq 0$, then $\rho \to 0$, $|f(z)-f(a)| \to 0$.

Example 3.3: Prove that $\sin z$ is continuous at every point except at $z=\infty$.
Solution: z=$x+iy$, a=$\xi+i\eta$. $|z-a|=\rho \leq r$.

$$|\sin z - \sin a| = 2|\cos\frac{z+a}{2}\sin\frac{z-a}{2}|.$$

Here we apply

$$|\cos z| = \frac{|e^{iz}+e^{-iz}|}{2} = \frac{|e^{ix}e^{-y}+e^{-ix}e^{y}|}{2} \leq \frac{|e^{ix}e^{-y}|+|e^{-ix}e^{y}|}{2} = \cosh y.$$

Hence

$$|\sin z - \sin a| \leq 2\cosh\left(\frac{y+\eta}{2}\right)|\sin\frac{z-a}{2}| \leq 2\cosh\left(y+\frac{\rho}{2}\right)|\sin\frac{z-a}{2}|.$$

One has used the fact of $\eta \leq y+\rho$ since $(x-\xi)^2+(y-\eta)^2=\rho^2$, that is $(y-\eta)^2 = \rho^2-(x-\xi)^2 \leq \rho^2$. Now z=$a+\rho e^{i\theta}$. Therefore

$$\begin{aligned} &\sin\left(\frac{z-a}{2}\right) = \sin\left(\frac{\rho\cos\theta + i\rho\sin\theta}{2}\right) \\ &= \sin\left(\frac{\rho\cos\theta}{2}\right)\cos\left(\frac{i\rho\sin\theta}{2}\right) + \cos\left(\frac{\rho\cos\theta}{2}\right)\sin\left(\frac{i\rho\sin\theta}{2}\right) \\ &= \sin\left(\frac{\rho\cos\theta}{2}\right)\cosh\left(\frac{\rho\sin\theta}{2}\right) + i\cos\left(\frac{\rho\cos\theta}{2}\right)\sinh\left(\frac{\rho\sin\theta}{2}\right). \end{aligned}$$

From $|x+iy| \leq |x|+|y|$ such that

$$|\sin\left(\frac{z-a}{2}\right)| \leq |\sin\left(\frac{\rho\cos\theta}{2}\right)||\cosh\left(\frac{\rho\sin\theta}{2}\right)|+|\cos\left(\frac{\rho\cos\theta}{2}\right)||\sinh\left(\frac{\rho\sin\theta}{2}\right)|.$$

We have

$$|\cosh\left(\frac{\rho\sin\theta}{2}\right)| = \cosh\left(\frac{\rho\sin\theta}{2}\right) \leq \cosh\left(\frac{\rho}{2}\right).$$

Since $-1 \leq \sin\theta \leq 1$ and $\cosh a \geq \cosh b > 0$ if $|a| \geq |b|$. On the other hand we also have

$$|\sinh\left(\frac{\rho\sin\theta}{2}\right)| = \sinh\left(|\frac{\rho\sin\theta}{2}|\right) \leq \sinh\left(\frac{\rho}{2}\right).$$

This is because $|\sinh x| = \sinh|x|$ and $\sinh a \geq \sinh b.0$ if $a > b > 0$. So we have

$$|\sin\left(\frac{z-a}{2}\right)| \leq \cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\rho\cos\theta}{2}\right)+\sinh\left(\frac{\rho}{2}\right)\cos\left(\frac{\rho\cos\theta}{2}\right)$$

$$\Longrightarrow |\sin z - \sin a| \leq 2\cosh\left(y+\frac{\rho}{2}\right)\cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\rho\cos\theta}{2}\right)+2\cosh\left(y+\frac{\rho}{2}\right)\sinh\left(\frac{\rho}{2}\right)\cos\left(\frac{\rho\cos\theta}{2}\right)$$

$$\Longrightarrow 2\cosh y\sin\left(\frac{\rho\cos\theta}{2}\right) \to 0.$$

when $\rho \to 0$ if $y \neq \infty$. Most of elementary functions are continuous ones. However the root and logarithms are exceptions. Let us look both cases.

Example 3.4: $f(z) = z^{1/3}$, please discuss its continuity.

Solution: Here we fist define the angle θ to be $0 \leq \theta \leq 2\pi$. Now let us consider the point above real axis: $z_1{=}re^{i\epsilon}$ here ϵ is a very small positive number. Nearby there is another point $z_2{=}re^{i(2\pi-\epsilon)}$. It is clear that the distance between z_1 and z_2 can be made as small as one wishes. Now let us investigate $w_1 = f(z_1) = (r)^{1/3}e^{i\epsilon/3}$ and $w_2 = f(z_2) = (r)^{1/3}e^{-i\epsilon}e^{i2\pi/3}$. Now we find that

$$\begin{aligned} |w_1 - w_2| = r^{1/3}|1-e^{-2i\epsilon}e^{i2\pi/3}| &= (r)^{1/3}\sqrt{2-2\cos\left(\frac{2\pi}{3}-2\epsilon\right)} \\ &\geq (r)^{1/3}\sqrt{2-2\cos\left(\frac{2\pi}{3}\right)} = (r)^{1/3}. \end{aligned}$$

Therefore we know $f(z)$ naively cannot be continuous function. However it will be very unnatural for such elementary function to be discontinuous function. So we introduce the devise called **Branch cut**.

Let us cut the complex plane along the real axis from 0 to $+\infty$. Now z_2 is no longer near z_1 because we "glue" the first plane with the second one which is also cut along $[0, +\infty]$. There will be z_3=$re^{i2\pi}e^{i\epsilon}$. Previously we identify z_3 to be z_1. Now we treat them as separate points. Similarly we can find z_4=$re^{i(4\pi-\epsilon)}$. z_2 and z_4 are also treated as separate points. Furthermore we "stick" the second "sheet" with the third one. Then z_5=$re^{i4\pi}e^{i\epsilon}$ is not near z_1 9at the first "sheet") and z_3 (at the second sheet). However z_6=$re^{i(6\pi-\epsilon)}$ is in the neighborhood of z_1 since we "stick" the third "sheet" to the first "sheet".

With such a devise we can claim $f(z)$=$z^{1/3}$ is continuous function. It is easy to see by writing the explicit values of w_i=$f(z_i)$:

$$\begin{aligned} w_1 &= (r)^{1/3}e^{i\epsilon/3}\ w_2 = (r)^{1/3}e^{-i\epsilon}e^{i2\pi/3},\ w_3 = (r)^{1/3}e^{i\epsilon/3}e^{i2\pi/3}\ w_4 = (r)^{1/3}e^{-i\epsilon}e^{i4\pi/3}, \\ w_5 &= (r)^{1/3}e^{i\epsilon/3}e^{i4\pi/3}\ w_6 = (r)^{1/3}e^{-i\epsilon}e^{i6\pi/3} = (r)^{1/3}e^{-i\epsilon}, \end{aligned}$$

Hence we notice w_6 is near w_1, w_2 is near w_3 and w_4 is near w_5. Remember z_1 is near z_6, z_2 is near z_3 and z_4 is near z_5. Consequently indeed $f(z)$=$z^{1/3}$ is continuous.

Example 3.5:$f(z) = \ln z$, please discuss its continuity.

Solution:

z=$re^{i\theta}$ here $0 \le \theta \le 2\pi$. w=$f(z)$=$\ln r + i\theta$.

Note that z_m=$z \cdot e^{i2m\pi}$ here m is an integer. It is obvious z=z_m but w_m=$f(z_m)$=$\ln r + i(\theta + m\pi)$. Therefore naively this is even not a function. However with the devise of branch cut we can define it as a continuous function.

Let us make a branch cut along the real axis $[0, \infty]$ and "stick" to the second "sheet" also along the same line. Then z_m will be located at the m-th sheet. By this way $f(z)$ is a well-defined function.

Next we can show that this function is continuous function. Suppose there are two points $z = re^{\theta}$ and $a = r'e^{i\theta'}$. z is near a and$|z - a| \le \rho$. Because two pints are in the same sheet, $\theta - \theta'$=ϕ. Now we have u=$\ln z$ and v=$\ln a$. Then $|u - v| = \sqrt{(\ln r - \ln r')^2 + \phi^2}$.

From $|z-a| \le |z|+|a|$ we have $r' \le r+\rho$ so $\ln\left(\frac{r'}{r}\right) \le \ln\left(1+\frac{\rho}{r}\right)$. Besides we have $\rho \ge r\phi$ so $\phi \le \frac{\rho}{r}$. Thus

$$|u-v| \le \left[\ln\left[1+\left(\frac{\rho}{r}\right)^2\right]+\left(\frac{\rho}{r}\right)^2\right]^{1/2} \to 0$$

as $\rho \to 0$. Sometimes we may need more branch cuts. Actually it is not unique to choose the branch cuts.

Example 3.6: Discuss the branch cut of $f(z)=\sqrt{1+z^2}$.

Solution:

$f(z)=\sqrt{(z+i)(z-i)}$. If we choose the branch cuts to run through its branch point upward along the imaginary axis, then we express any point z in the complex plane as

$$z = i+\rho_1 e^{i\phi_1} = -i+\rho e^{i\phi_2}. \quad -\frac{3\pi}{2} \le \phi_{1,2} \le \frac{\pi}{2},$$

Then the branch cut will be $[+i, +i\infty]$ and $[-i, +i\infty]$. Besides

$$w = f(z) = \sqrt{\rho_1\rho_2}e^{i(\phi_1+\phi_2)/2}.$$

Naively one may think the continuity of this function will be lost along the imaginary axis. The situation is more complex. Choose two points: $z_1 = \epsilon+ia$ and $z_2 = -\epsilon+ia$. ϵ is an arbitrary small number. When $a>1$ then for z_1 we have $\phi_1=-\frac{3\pi}{2}+\eta_1$ and $\phi_2=-\frac{3\pi}{2}+\eta_2$ here $\eta_{1,2}$ are arbitrary small angles. Then $w_1 = f(z_1) = \sqrt{\rho_1\rho_2}e^{-i3\pi/2}e^{i(\eta_1+\eta_2)/2}=i\sqrt{\rho_1\rho_2}e^{i(\eta_1+\eta_2)/2}$. For z_2 we have $\phi_1=\frac{\pi}{2}-\eta_1$ and $\phi_2=\frac{\pi}{2}-\eta_2$. $w_2 = f(z_2) = \sqrt{\rho_1\rho_2}e^{i\pi/2}e^{-i(\eta_1+\eta_2)/2}=i\sqrt{\rho_1\rho_2}e^{-i(\eta_1+\eta_2)/2}$. So w_1 is near w_2. The function is continuous there.

On the other hand, if $-1 \le a \le 1$ then for z_1 we have $\phi_1=-\frac{\pi}{2}-\eta_1$ and $\phi_2=-\frac{3\pi}{2}+\eta_2$ here $\eta_{1,2}$ are arbitrary small angles. Then $w_1=f(z_1)=\sqrt{\rho_1\rho_2}e^{-i\pi}e^{i(-\eta_1+\eta_2)/2}=-\sqrt{\rho_1\rho_2}e^{i(-\eta_1+\eta_2)/2}$. For z_2 we have $\phi_1=\frac{-\pi}{2}+\eta_1$ and $\phi_2=\frac{\pi}{2}-\eta_2$. $w_2 = f(z_2) = \sqrt{\rho_1\rho_2}e^{i(\eta_1-\eta_2)/2}$. It is clear the function is discontinuous there.

Last when $a \le -1$, then for z_1 we have $\phi_1=-\frac{\pi}{2}-\eta_1$ and $\phi_2=-\frac{\pi}{2}-\eta_2$ here $\eta_{1,2}$ are arbitrary small angles. Then $w_1 = f(z_1) = \sqrt{\rho_1\rho_2}e^{-i\pi/2}e^{-i(\eta_1+\eta_2)/2}=-i\sqrt{\rho_1\rho_2}e^{-i(\eta_1+\eta_2)/2}$. For z_2 we have $\phi_1=-\frac{\pi}{2}+\eta_1$ and $\phi_2=-\frac{\pi}{2}+\eta_2$. $w_2 = f(z_2) = \sqrt{\rho_1\rho_2}e^{-i\pi/2}e^{i(\eta_1+\eta_2)/2}=-i\sqrt{\rho_1\rho_2}e^{i(\eta_1+\eta_2)/2}$. So w_1 is near w_2. The function is continuous there.

Thus the discontinuity only occurs in $[-i, +i]$.

Exercise 3.1: If $0 \leq \phi_{1,2} \leq 2\pi$, please discuss its branch cuts and the discontinuity of the function.

Home Work 3.2: Describe how to construct Riemann surfaces to make the followings mapping as continuous:

(a)$f(z)=\left(\frac{z+1}{z-1}\right)^{1/3}$. (b)$f(z)=\sqrt{z+1}\ln z$. (c) $f(z)=(1+\sqrt{z})^{1/2}$.

III. STEREOGRAPHIC PROJECTION

Sometimes we need pay special attention to he point at infinity. The best way to study it is by the stereographic projection. Imagine there is a sphere: $\mathcal{S}=x_1^2+x_2^2+(x_3-\frac{1}{2})^2=\frac{1}{4}$. We notice the point N=(0,0,1) is the north pole. S=(0,0,0) is the south pole. We can set up a mapping between x_1-x_2 plane and S. Assume there is a point $A=(a,b,0)$ in the x_1-x_2 plane. The line $\overline{NA}$ intersects $\mathcal{S}$ at the point B. $B=\left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right)$. Besides we notice S which is at $\mathcal{S}$ and $\overline{NS}$ intersects $\mathcal{S}$ at S.

Now we can identify x_1-x_2 plane as complex plane and $z=a+ib$. Therefore each point in the complex plane corresponds to one point at $\mathcal{S}$. Interestingly one notices that when $|z| \to \infty$, the corresponding point at $\mathcal{S}$ is N. This is very convenient way to "imagine" the point at infinity in the complex plane.

Home Work 3.3: Please find the spherical image of the circle $|z-a|=\rho$ in the complex plane.

IV. CAUCHY THEOREM,TAYLOR SERIES,LAURENT SERIES

How to define a differential of a complex function $f(z)=u(x,y)+iv(x,y)$ $(z=x+iy)$? The straightforward answer is to take the following definition:

$$\frac{df}{dz}=\lim_{a\to 0}\frac{f(z+a)-f(z)}{a}.$$

Here $a=a_x+ia_y$ is an arbitrary small complex number. To make sure by this way one always obtains the same result no matter which path one chooses, one finds that a complex function is differentiable if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

This relation is called **Cauchy − RiemannRelation**.

Exercise 3.2 Prove that a complex function is differentiable if and only if it satisfies Cauchy-Riemann relation.

Solution:

First we prove that Cauchy-Riemann relation is necessary condition for differentiability. We can choose the $dz = dx$

$$\begin{aligned}\frac{df}{dz} &= \frac{f(x + dx + iy) - f(x, y)}{dx} \\ &= \frac{u(x + dx, y) + iv(x + dx, y) - u(x, y) - iv(x, y)}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.\end{aligned}$$

On the other hand we can choose $dz{=}idy$

$$\begin{aligned}\frac{df}{dz} &= \frac{f(x + iy + idy) - f(x, y)}{dx} \\ &= \frac{u(x, y + dy) + iv(x, y + dy) - u(x, y) - iv(x, y)}{idy} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.\end{aligned}$$

Comparing the two results we know it is necessary for f to satisfy Cauchy-Riemann relation to be differentiable. Next let us prove any function satisfy Cauchy-Riemann relation must be differentiable. We notice that $a_x + ia_y = r(\cos\theta + i\sin\theta)$. Then

$$\begin{aligned}&\lim_{r\to 0} \frac{u(x + a_x, y + a_y) + iv(x + a_x, y + a_y) - u(x, y) - iv(x, y)}{a_x + ia_y} \\ &= e^{-i\theta}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta + i\frac{\partial v}{\partial x}\cos\theta + i\frac{\partial v}{\partial y}\sin\theta\right)\end{aligned}$$

This is because r is canceled and $\frac{1}{\cos\theta + i\sin\theta} = e^{-i\theta}$. So that

$$\begin{aligned}&= \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial u}{\partial y}\sin\theta\cos\theta + i\frac{\partial v}{\partial x}\cos^2\theta + i\frac{\partial v}{\partial y}\sin\theta\cos\theta \\ &- i\frac{\partial u}{\partial x}\cos\theta\sin\theta - i\frac{\partial u}{\partial y}\sin^2\theta + \frac{\partial v}{\partial x}\cos\theta\sin\theta + \frac{\partial v}{\partial y}\sin^2\theta\end{aligned}$$

Applying Cauchy-Riemann relation it becomes

$$\begin{aligned} &= \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial u}{\partial y}\sin\theta\cos\theta - i\frac{\partial u}{\partial y}\cos^2\theta + i\frac{\partial u}{\partial x}\sin\theta\cos\theta \\ &- i\frac{\partial u}{\partial x}\cos\theta\sin\theta - i\frac{\partial u}{\partial y}\sin^2\theta - \frac{\partial u}{\partial y}\cos\theta\sin\theta + \frac{\partial u}{\partial x}\sin^2\theta \\ &= \frac{\partial u}{\partial x}(\cos^2\theta + i\sin\theta\cos\theta - i\cos\theta\sin\theta + \sin^2\theta) \\ &+ \frac{\partial u}{\partial y}(\sin\theta\cos\theta - i\cos^2\theta - i\sin^2\theta - \cos\theta\sin\theta) \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \end{aligned}$$

Hence we find the above expression is independent of θ which means taking differential in any direction will reach the same result. Hence $f(z)$ is differentiable.

Home work 3.4: If $f(z))$ is analytic function then $\frac{\partial f}{\partial z^*}$=0.

Analytic Complex Functions

A complex function which is differentiable at $z = a$ and within a neighborhood of $z = a$ is said to be **analytic** at $z = a$. Here we introduce the most important theorem in complex analysis.

Cauchy Theorem:

If $f(z)$ is analytic in and on C then

$$\oint_C f(z)dz = 0.$$

Example 3.7: Prove the Cauchy theorem.

Solution:

$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2}(u+iv)(dx+idy) = \int_{z_1}^{z_2}(udx - vdy) + i(udy + vdx)$. Set $\vec{A} = (u, -v, 0)$ and $\vec{B} = (v, u, 0)$. $\nabla \times \vec{A} \cdot \hat{n} = -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $\nabla \times \vec{B} \cdot \hat{n} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$. Thus we have

$$\oint_C f(z)dz = \oint_C \vec{A}(x,y)\cdot d\vec{l} + i\oint_C \vec{B}(x,y)\cdot d\vec{l} = \int_S \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + i\int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0$$

There are many applications of Cauchy theorem. One of them is Cauchy formula:

Cauchy Formula

If $z = a$ lies within C and $f(z)$ is analytic in and on C, one has

$$\oint \frac{f(z)}{z-a}dz = 2\pi i f(a).$$

Otherwise, the integral is zero.

Proof: Let Γ is a circle with the radius ρ with $z = a$ as the center.

Making a path C'=$C + B_1 + B_2 + \Gamma$ with B_1 the path connecting C and Γ, B_2 a path very close to B_1 with the opposite direction. From Cauchy Theorem, since $F(z) = \frac{f(z)}{z-a}$ is analytic on and in C', such that

$$0 = \oint_{C'} \frac{f(z)}{z-a} dz = \left(\oint_{C} + \oint_{\Gamma, Clockwise} + \int_{B_1} + \int_{B2} \right) \frac{f(z)}{z-a} dz$$

Since $\int_{B_1} + \int_{B_2} = 0$ one has

$$\oint_C \frac{f(z)}{z-a} dz = \oint_\Gamma \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} \rho e^{i\theta} i d\theta = i \int_0^{2\pi} f(a+\rho e^{i\theta}) d\theta.$$

Taking the limit $\rho \to 0$

$$\oint_C \frac{f(z)}{z-a} dz = \lim_{\rho \to 0} i \int_0^{2\pi} f(a+\rho e^{i\theta}) d\theta = 2\pi i f(a).$$

Home Work 3.5: Evaluate $\oint_{|z|=1} \frac{\cos^3 z}{z} dz$. Applying Cauchy formula one can approve the following important result:

Exercise 3.3: Apply induction to prove that

$$\oint \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a).$$

Solution:

Assume $\oint \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$ now we need prove $\oint \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{(n)!} f^{(n)}(a)$. It should proceed as

$$\begin{aligned} f^{(n)}(a) &= \lim_{h \to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} \\ &= \lim_{h \to 0} \frac{(n-1)!}{2\pi i h} \left(\oint \frac{f(z)}{(z-a-h)^n} dz - \oint \frac{f(z)}{(z-a)^n} dz \right) \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{(z-a-h)^n} - \frac{1}{(z-a)^n} &= \frac{1}{(z-a)^n} \left(\frac{1}{\left(1 - \frac{h}{z-a}\right)^n} - 1 \right) \\ &= \frac{1}{(z-a)^n} \left[\left(1 + \sum_{k=1}^{\infty} \left(\frac{h}{z-a} \right)^k \right)^n - 1 \right]. \\ &= \frac{nh}{(z-a)^{n+1}} + \mathcal{O}(h^2). \end{aligned}$$

Hence

$$f^{(n)}(a) = \lim_{h\to 0} \frac{(n-1)!}{2\pi i h} \oint \frac{nhf(z)}{(z-a)^{n+1}}dz + \mathcal{O}(h)$$
$$= \frac{n(n-1)!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}dz = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}dz.$$

Since we have proved the case of $n = 1$, therefore we have proved for the case of arbitrary integer n.

Home Work 3.6: Legendre polynomials are defined as $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. Please show that

$$P_n(z) = \frac{1}{2^n}\frac{1}{2\pi i} \oint \frac{(t^2-1)^n}{(t-z)^{n+1}}dt.$$

Furthermore if the contour is chosen to be a circle of radius $\sqrt{z^2-1}$ centered at t=z, Please show that

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} (z + \sqrt{z^2-1}\cos\phi)^n d\phi.$$

From Cauchy integral formula one can prove many interesting facts. They are left for students to prove.

Home Work 3.7: (**Cauchy's inequality**): If $f(z)$ is analytic and bound inside and on C;$|z-z_0| \le r$ and $|f(z)| < M$ on C . Please show that $|f^{(n)}(z=z_0)| \le \frac{n!M}{r^n}$.

Home Work 3.8: (**Liouville's theorem**) If $f(z)$ is analytic in the entire complex plane and is bounded (i.e. $|f(z)| < M$ for some constant M). Please prove that $f(z)$ must be a constant.

Home Work 3.9: (**Fundamental theorem of algebra**) Please prove that every polynomial equation of degree $n \ge 1$ with complex coefficients has at least one root.

Home Work 3.10:(**Maximum modulus theorem**) Let $f(z)$ be analytic on a closed region $R : |z-z_0| \le \rho$ with boundary $C : |z-z_0| = \rho$ and let M be the maximum value assumed by $|f(z)|$ in R. Then, if $f(z)$ is not identically equal to a constant, the maximum value M of $|f(z)|$ occurs on the boundary C. In addition, at all points on the interior of R, $|f(z)| < M$.

V. TAYLOR SERIES AND LAURENT SERIES

Taylor Series:

If $f(z)$ is analytic in a region $|z-a| \leq \rho$. Then

$$f(z) = \sum_{n=0}^{n=\infty} \frac{(z-a)^n}{n!} \frac{d^n f(z)}{dz^n}|_{z=a}.$$

The series is uniformly convergent within $|z-a| \leq \rho$.

Proof:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\Gamma \frac{\xi}{\xi - z} d\xi = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - a} \left(\frac{1}{1 - \frac{z-a}{\xi-a}} \right) d\xi = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - a} \sum_{n=0}^{n=\infty} \left(\frac{z-a}{\xi-a} \right)^n d\xi \\ &= \frac{1}{2\pi i} \sum_{n=0}^{n=\infty} (z-a)^n \oint_\Gamma \frac{f(\xi)}{(\xi-a)^{n+1}} \sum_{n=0} d\xi = \sum_{n=0}^{n=\infty} \frac{(z-a)^n}{n!} \frac{d^n f(z)}{dz^n}|_{z=a}. \end{aligned}$$

Laurent Series:

If $f(z)$ is analytic in an annular region $\rho_1 < |z-a| < \rho_2$. Then

$$f(z) = \sum_{n=-\infty}^{n=\infty} c_n (z-a)^n. \ c_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi$$

Exercise 3.4: Prove Laurent series. C=$C_1 + C_2 + \Gamma$+cross cuts.

$$\oint_C \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi + \oint_{\Gamma, clockwise} \frac{f(\xi)}{\xi - z} d\xi + \oint_{C_1, clockwise} \frac{f(\xi)}{\xi - z} d\xi + \oint_{crosscuts} \frac{f(\xi)}{\xi - z} d\xi = 0.$$

$$\oint_\Gamma \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi - \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi.$$

For C_1:$|z-a| > |\xi - a|$.

$$\frac{1}{\xi - z} = \frac{-1}{z-a} \frac{1}{\left(1 - \frac{\xi-a}{z-a}\right)} = \frac{-1}{z-a} \sum_{n=0}^{\infty} \left(\frac{\xi-a}{z-a} \right)^n.$$

For C_2:$|z-a| < |\xi - a|$.

$$\frac{1}{\xi - z} = \frac{1}{\xi-a} \frac{1}{\left(1 - \frac{z-a}{\xi-a}\right)} = \frac{1}{\xi-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi-a} \right)^n.$$

Therefore

$$\oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=0}^{\infty} \frac{-1}{(z-a)^{n+1}} \oint_{C_1} f(\xi)(\xi - a)^n d\xi.$$

$$\oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=0}^{\infty} (z-a)^n \oint_{C_2} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi.$$

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=-\infty}^{n=\infty} c_n (z-a)^n.$$

$$c_n = \frac{1}{2\pi i} \oint_{C} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi.$$

Home Work 3.11: Determine the Taylor or Laurent series for each of the following functions in the neighborhood of the point specified.
(a)$\frac{\cos z}{z-1}$ at $z = 1$. (b) $\frac{\ln z}{z-1}$ at $z = 1$ (c) $\tan^{-1} z$ at $z = 0$. (d) $\frac{e^z}{z^2+1}$ at $z = 0$.

VI. 2-D HARMONIC FUNCTIONS

The real and imaginary parts of an analytic function $f(z)$=$u + iv$ satisfies Laplace's equation in two dimensions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \to \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0.$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \to \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \nabla^2 v = 0.$$

Therefore we can apply result of previous discussion to the 2-D harmonic real functions. For example for the proof we know

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Taking the real part of both side we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

This is very interesting result. It tells you that the values of the boundary of a harmonic function determine the value of the function at center. We can generalize this result to the value of any point inside the region. It is called Poisson's formula.

Poisson's formula

If $u(z = x + iy)$ is a real harmonic function in $|z| \leq R$. The transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + a^*\zeta}, \quad \zeta = \frac{R(z - a)}{R^2 - a^*z}.$$

It maps $|\zeta| \leq 1$ to $|z| \leq R$ and $\zeta = 0$ to $z = a$. a is a complex number inside $|z| \leq R$.

Exercise 3.5 Please show that the imagine of $|\zeta|$=1 is $|z|$=R.

Here z=re^{θ} and ζ=$\rho e^{i\phi}$. Apply the mean property of harmonic function we have

$$u(a) = \frac{1}{2\pi}\int_{\rho=1} u(S(\zeta))d\phi.$$

For the points at $|\zeta|$=1, dz=$ire^{i\theta}d\theta$=$izd\theta$

$$\begin{aligned}
d\phi &= -i\frac{d\zeta}{\zeta}, d\zeta = \frac{R}{R^2 - a^*z}dz + \frac{R(z-a)a^*}{(R^2 - a^*z)^2}dz, \\
\frac{d\zeta}{\zeta} &= \frac{1}{z-a}dz + \frac{a^*}{R^2 - a^*z}dz = \left(\frac{z}{z-a} + \frac{a^*z}{R^2 - a^*z}\right)dz. \\
d\phi &= -i\left(\frac{1}{z-a} + \frac{a^*}{R^2 - a^*z}\right)dz = \left(\frac{z}{z-a} + \frac{a^*z}{R^2 - a^*z}\right)d\theta
\end{aligned}$$

For the points at $|z|$=R, zz^*=R^2. Hence

$$\left(\frac{z}{z-a} + \frac{a^*z}{R^2 - a^*z}\right) = \left(\frac{z}{z-a} + \frac{a^*z}{zz^* - a^*z}\right) = \left(\frac{z}{z-a} + \frac{a^*}{z^* - a^*}\right) = \frac{R^2 - |a|^2}{|z-a|^2}.$$

At the end we have

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}u(z)d\theta.$$

In polar coordinate we have

$$u(re^{i\xi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\xi - \theta) + r^2}u(Re^{i\theta})d\theta.$$

So far we assume that u is harmonic in the closed disk $|z| \leq R$. Actually the result is still valid if u is only harmonic in the open disk $|z| < R$ and continuous at the boundary $|z|$=R. The proof is simple. Assume $u(z)$ is harmonic in $|z| < R$ and continuous at $|z| = R$. Suppose $0 < \alpha < 1$ then $u(\alpha z)$ is harmonic in $|z| \leq R$ so that

$$u(\alpha a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}u(\alpha z)d\theta.$$

Now we let α approaches 1. Since $u(z)$ is uniformly continuous on $|z| \leq R$ it is true that $u(\alpha z) \to u(z)$ uniformly for $|z| = R$. So the above result holds even $u(z)$ is not harmonic at

$|z|$=R.

Exercise 3.6 Please use the mean property of harmonic function to prove $\int_0^\pi \ln\sin\theta d\theta$=$-\pi\ln 2$..

Exercise 3.7 Please prove $\int_0^\pi \ln|1-e^{i\eta}|d\theta$=0. η is an arbitrary constant.

Jensen's Formula:

If $f(z)$ is an analytic function then $\ln|f(z)|$ is harmonic except at the zeros of $f(z)$. Therefore if $f(z)$ is analytic and has no zeros in $|z|\le R$, then

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta,$$

This is called Jensen's formula. Actually this result is still applicable even $f(z)$ has zeros at $|z| = R$. For example, if there is one zero z_1=$Re^{i\eta}$ such that $f(z_1)$=0. Then we construct $g(z)$=$\frac{f(z)}{z-z_1}$. Then we have

$$\ln|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(Re^{i\theta})|d\theta.$$

Since $\ln|g(0)|$=$\ln|f(0)| - \ln R$ and

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \ln|g(Re^{i\theta})|d\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \ln|Re^{i\theta}-Re^{i\eta}|d\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \ln|e^{i\theta}-Re^{i\eta}|d\theta - \ln R.
\end{aligned}$$

From Exercise 3.7 we know $\int_0^\pi \ln|1-e^{i\eta}|d\theta$=0. Hence

$$\int_0^\pi \ln|e^{i\theta}-e^{i\theta_0}|d\theta = \int_0^\pi (\ln|e^{i\theta}| + \ln|1-e^{i(\theta-\theta_0)}|)d\theta = 0.$$

So that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|g(Re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta - \ln R.$$

At the end we have

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta.$$

Home Work 3.12: If $f(z)$ has zeros z=$z_1, z_2, z_3, ...z_n$ in $|z| < \rho$. Note that $f(0) \neq 0$. Please show that

$$\ln|f(0)| = -\sum_{i=1}^{n} \ln\left(\frac{\rho}{|z_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\theta})|d\theta.$$

If $f(z)$ is analytic and $g(z)$ is also analytic, we know $f(g(z))$ is also analytic. If the boundary condition of Laplace equation can be mapped to simpler one by $f(z)$, and with the simpler boundary condition one obtain the solution which is real part(or imaginary) part of an analytic function $g(z)$, then our solution is the real(or imaginary) of $f^{-1}(g(z))$. To find the suitable mapping needs some skill. One important fact about the mapping is the following:

Conformal mapping:

If $f(z)$ is analytic at $z = a$ and $f^{(1)}(z) \neq 0$ then $f(z)$ is conformal mapping.

Proof:

C_1=$g(z) = Constant$ and C_2=$h(z) = Constant$ intersects at $z = z_0$. The tangent vector at $z = z_0$ along C_1 is dz_1. The tangent vector along C_2 is dz_2. Under the mapping: $w = f(z)$ The curve C_1 is mapped to C_1'=$f(g(z)) = Constant$ and C_2 is mapped to C_2'=$f(h(z))$. The tangent vectors along C_1' and C_2' are dw_1 and dw_2. The angle between dz_1 and dz_2 is $\theta = arg(dz_2/dz_1)$. Similarly the angle between dw_1 and dw_2 is $\phi = arg(dw_1/dw_2)$. Since $dw_1 = \frac{df}{dz}|_{z=z_0}dz_1$ and $dw_2 = \frac{df}{dz}|_{z=z_0}dz_2$. Therefore ϕ=$arg(dw_1/dw_2)$=$arg\left(\frac{\frac{df}{dz}|_{z=z_0}dz_1}{\frac{df}{dz}|_{z=z_0}dz_2}\right)$ =$arg(dz_2/dz_1)$=θ. Therefore analytic complex function regarded as a mapping then this mapping is conformal mapping.

In particular we are interested in this form $f(z)$=$\frac{az+b}{cz+d}$ because this type of mapping can map a circle or a line to another circle or line. To map the boundary condition to simple one, this type of mapping is very useful.

Example 3.9: Apply the transformation $w = f(z) = 2R^2/z$ to find the potential outside the metal cylinder which has radius R and electric potential V. The x-axis is at zero potential.

Solution:

When z=$re^{i\theta}$, one has

$$w = u + iv = \frac{2R^2}{r}e^{-i\theta},\ u = \frac{2R^2}{r}\cos\theta, v = -\frac{2R^2}{r}\sin\theta.$$

Now the line $v = v_0$ in the $w-$plane is the imagine of the following curve:

$$-\frac{2R^2}{r}\sin\theta \;=\; v_0, \;\; r = -\frac{-2R^2}{v_0}\sin\theta.$$
$$x \;=\; r\cos\theta = \frac{-2a}{v_0}\sin\theta\cos\theta = \frac{-R^2}{v_0}\sin 2\theta,$$
$$y \;=\; r\sin\theta = \frac{-2a}{v_0}\sin^2\theta = \frac{-R^2}{v_0}(1-\cos 2\theta).$$

Namely

$$x^2 + \left(y + \frac{R^2}{v_0}\right)^2 = \left(\frac{R^2}{v_0}\right)^2.$$

Therefore $x^2 + (y-R)^2 = R^2$ is mapped to $v = -R$ if we choose $v_0 = -R$. The line $y = 0$ is mapped to $v = 0$. In the $w-$ plane the potential is $\phi = -V\frac{v}{R} = Re\left(-V\frac{w}{iR}\right) = Re\Phi(w)$. Now $\Phi(z) = i\frac{V}{R}\frac{2R^2}{z} = \frac{2RV}{r}(\sin\theta + i\cos\theta)$. Taking the real part of $\Phi(z)$ we have $\phi = \frac{2RV}{r}\sin\theta$.

VII. POLES AND ZEROS

If $f(z)$ is analytic in the neighborhood of a point $z = a$ but not at $z = a$ then a is an **isolated singularity** of $f(z)$. There are three cases

1. $|f(z)| \to \infty$ as $z \to a$. It is called **pole**.

2. $|f(z)|$ is bound as $z \to a$. It is called **removable**.

3. The value of $\lim_{z\to a}|f(a)|$ depends on the path of the process of taking limit. It is called **essential singlarity**.

Home Work 3.13: Find the singularities of the following functions and determine what kind of singularities they are. (a) $\ln\frac{e^z}{z} - \sin\frac{1}{z}$. (b) $\frac{\tanh z}{z}$. (c) $\ln(1+z^2)$.

Example 3.10: Explain $\tan\left(\frac{1}{z}\right)$ has a singularity at $z = 0$ which is not isolated singularity.

Solution:

$\tan w$ has poles at $w{=}\frac{n\pi}{2}$. Therefore $z_n{=}\frac{2}{n\pi}$ is singularity. Now we find inside $|z| < \rho$

there are infinite many singularities because we find that for arbitrary small ϵ, as long as $m > n_0 = \frac{1}{\epsilon}$ then $z_m = \frac{2}{m\pi} < \frac{2}{n_0\pi} < \frac{2}{\pi}\epsilon < \epsilon$. Those z_m are in the region $|z| < \epsilon$. Therefore we show that z=0 is not an isolated singularity.

Order of pole:

If $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-m}^{n=\infty} c_n (z-a)^n.$$

then the pole $z = a$ is called the pole of order m.