Lecture V: Fourier Transforms

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This lecture introduces the basic elements of Fourier transforms.

I. FOURIER SERIES

Any moderately well-behaved function $f(x)$ which satisfying $f(x)$=$f(x+2\pi)$ can be expressed as sum of a sum of sines and cosines:

$$f(x)=\sum_{n=0}^{\infty}(a_n\sin nx+b_n\cos nx).$$

Another form from is

$$f(x)=\sum_{n=-\infty}^{n=\infty}c_ne^{inx}.$$

Here

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx dx,\; b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx,\;\; n\geq 1,\\
b_0 &= \frac{1}{2\pi}\int_0^{2\pi}f(x)dx\\
c_n &= \frac{1}{2\pi}\int_0^{2\pi}f(x)e^{-inx}dx
\end{aligned}$$

By changing variable we can now apply this result to any period function with the period L, namely $f(x)=f(x+L)$:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}(a_n\sin\frac{2n\pi x}{L}+b_n\cos\frac{2\pi nx}{L}).\\
a_n &= \frac{2}{L}\int_0^{L}f(x)\sin\left(\frac{2n\pi x}{L}\right)dx,\; b_n=\frac{2}{L}\int_0^{L}f(x)\cos\left(\frac{2n\pi x}{L}\right)dx,\;\; n\geq 1,\\
b_0 &= \frac{1}{L}\int_0^{L}f(x)dx
\end{aligned}$$

Another form from is

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{i2n\pi x/L}.$$
$$c_n = \int_0^L f(x)e^{-i2n\pi x/L}dx.$$

Sometimes $f(x)$ is defined within $-L \le x \le L$. Then we have

$$f(x) = \sum_{n=0}^{\infty}(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{\pi n x}{L}).$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ \ n \ge 1,$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

Example 5.1:

Obtain the Fourier series representation of the function: $f(x) = x^2, -\pi \le x \le x$. and use it to show that (a) $\frac{\pi^2}{6}{=}\sum_{n=1}^{\infty}\frac{1}{n^2}$. (b)$\frac{\pi^2}{12}{=}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Solution:

$$b_0 = \frac{1}{2\pi}\int_{\pi}^{\pi} x^2 dx = \frac{\pi^2}{3},$$
$$b_n = \frac{1}{\pi}\int_{\pi}^{\pi} x^2\cos nx dx = \frac{1}{\pi}\int_{\pi}^{\pi} x^2\frac{1}{n}d\sin nx = x^2\frac{\sin nx}{n}|_{x=-\pi}^{x=\pi} - \frac{1}{n}\int \sin nx dx^2$$
$$= -\frac{2}{n}\int_{x=-\pi}^{x=\pi} x\sin nx dx = \frac{2}{n^2}\int_{x=-\pi}^{x=\pi} x d\cos nx = \frac{2}{n^2}x\cos nx|_{x=-\pi}^{x=\pi} - \frac{2}{n^2}\int_{x=-\pi}^{x=\pi}\cos nx dx$$
$$= \frac{2}{n^2}((\pi)(-1)^n - (-\pi)(-1)^n) - \frac{2}{n^3}(\sin(n\pi) - \sin(-n\pi))$$
$$= \frac{4\pi}{n^2}(-1)^n.$$

So that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$.

(a) At $x = \pi$: $\pi^2{=}\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Longrightarrow \frac{\pi^2}{6}{=}\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(b) At $x = 0$: $0{=}\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Longrightarrow \frac{\pi^2}{12}{=}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$.

Example 5.2: Show that $\sum_{n=1}^{\infty}\frac{\sin nx}{n}{=}\frac{1}{2}(\pi - x), 0 \le x \le \pi; \frac{-1}{2}(\pi + x), -\pi \le x \le 0$.

Solution:

$f(x)=\frac{1}{2}(\pi - x), 0 \le x \le \pi; \frac{-1}{2}(\pi + x), -\pi \le x \le 0$. Look at its Fourier Series representation:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\frac{-1}{2}(\pi + x)dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\frac{1}{2}(\pi - x)dx \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} x\sin nx dx - \frac{1}{2}\int_{-\pi}^{0}\sin nx dx + \frac{1}{2}\int_{0}^{\pi}\sin nx dx \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} x\sin nx dx - \frac{1}{2n}[-\cos nx]_{x=-\pi}^{x=0} + \frac{1}{2n}[-\cos nx]_{x=0}^{x=\pi} \\
&= \frac{1}{2\pi n}\int_{-\pi}^{\pi} xd\cos nx - \frac{1}{2n}[-\cos 0 + \cos(-n\pi)] + \frac{1}{2n}[-\cos(n\pi) + \cos 0] \\
&= \frac{1}{2\pi n}[x\cos nx]|_{x=-\pi}^{x=\pi} - \frac{1}{2n\pi}\int_{-\pi}^{\pi}\cos nx dx - \frac{1}{2n}[-1 + (-1)^n] + \frac{1}{2n}[-(-1)^n + 1] \\
&= \frac{1}{2\pi n}[\pi(-1)^n - (-\pi)(-1)^n] + \frac{1}{2\pi n^2}[\sin nx]|_{x=-\pi}^{x=\pi} + \frac{1}{n} - \frac{1}{n}(-1)^n \\
&= \frac{1}{n}[(-1)^n] + \frac{1}{2\pi n^2}[\sin(n\pi) - \sin(-n\pi)] + \frac{1}{n} - \frac{1}{n}(-1)^n \\
&= \frac{1}{n}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\frac{-1}{2}(\pi + x)dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\frac{1}{2}(\pi - x)dx \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} x\cos nx dx - \frac{1}{2}\int_{-\pi}^{0}\cos nx dx + \frac{1}{2}\int_{0}^{\pi}\cos nx dx \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} x\cos nx dx - \frac{1}{2n}[\sin nx]_{x=-\pi}^{x=0} + \frac{1}{2n}[\sin nx]_{x=0}^{x=\pi} \\
&= \frac{-1}{2\pi n}\int_{-\pi}^{\pi} xd\sin nx - \frac{1}{2n}[\sin n\pi - 0] + \frac{1}{2n}[\sin(n\pi) - \sin 0] \\
&= \frac{-1}{2\pi n}[x\sin nx]|_{x=-\pi}^{x=\pi} + \frac{1}{2n\pi}\int_{-\pi}^{\pi}\sin nx dx \\
&= \frac{1}{2\pi n^2}[-\cos nx]|_{x=-\pi}^{x=\pi} \\
&= \frac{1}{2\pi n^2}[-\cos(-n\pi) + \cos(-n\pi)] = 0
\end{aligned}
$$

Exercise 5.1: By using expansion of $-\ln(1-z)$ to show that $\sum_{n=1}\frac{\cos nx}{n} = -\ln\left(2\sin\frac{x}{2}\right)$ for $0 < x < 2\pi$.

Home Work 5.1: Using the Fourier transform of $f(x)=x^4$, $-\pi \le x \le \pi$ to derive the following results: (a) $\frac{1}{n^4}=\frac{\pi^4}{90}$. (b) $\frac{(-1)^{n+1}}{n^4}=\frac{7\pi^4}{720}$.

Home Work 5.2: Using the Fourier transform of $f(x)$=x when $0 \le x \le \pi$ and $f(x)$=0 when $-\pi \le x \le 0$ to derive the following result: $\frac{1}{(2n+1)^2}$=$\frac{\pi^2}{8}$.

II. APPLICATION OF FOURIER SERIES

Example 5.3: Find the solution of $v^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ with the boundary conditions: $y(x=0,t) = y(x=L,t)$=0 and the initial condition:

$$y(x,t=0) = \frac{2hx}{L}, 0 \le x \le L/2; \frac{2h(L-x)}{L}, L/2 \le x \le L$$

and

$$\frac{\partial y}{\partial t}(x,t=0) = 0$$

Solution:

Let $y(x,t)$=$X(x)T(t)$. We have $\frac{X''}{X}$=$\frac{T''}{v^2T}$=$-k^2$. Then $X(x)$=$A\sin kx + B\cos kx$ and $T(t)$=$C\cos kvt$+$D\sin kvt$. To satisfy the boundary conditions. We have B=0 and $\sin kL$=0. Thus k_n=$\frac{n\pi}{L}$. Therefore we have

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi vt}{L}\right) + b_n \cos\left(\frac{n\pi vt}{L}\right)\right].$$

To satisfy the initial condition: $\frac{\partial y}{\partial t}(x,t=0) = 0$ one has a_n=0. Now we need evaluate b_n. Since $y(x,t=0)$=$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$. So

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^L dxy(x,0)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{4h}{L^2}\left(\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\ &= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2} \end{aligned}$$

Home Work 5.3: Please work out the integral in Example 5.3

Home Work 5.4: Adding a term in the Equation of Example 5.3: $v^2\frac{\partial^2 y}{\partial x^2} - k\frac{\partial y}{\partial t}$=$\frac{\partial^2 y}{\partial t^2}$

III. FOURIER TRANSFORMATION

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L},\;\; a_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx.$$

$L \to \infty$, $n\pi/L \to 0$. Defining $k = \frac{n\pi}{L}$ and $\Delta k = \frac{\pi}{L}$

$$\begin{aligned} f(x) &= \lim_{L\to L}\sum_{n=-\infty}^{\infty} a_n e^{ikx}\Delta k \cdot \frac{2L}{\pi} = \frac{1}{\sqrt{2\pi}}\lim_{L\to\infty}\sum_{n=-\infty}^{\infty} a(k)e^{ikx}\Delta k \cdot \frac{2L}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}}\lim_{L\to L}\sum_{n=-\infty}^{\infty}\left(\frac{2L}{\sqrt{2\pi}}a(k)\right)e^{ikx}\Delta k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkF(k)e^{ikx}dk. \end{aligned}$$

Here

$$\begin{aligned} F(k) &= \lim_{L\to\infty}\left(\frac{2L}{\sqrt{2\pi}}a(k)\right) = \lim_{L\to\infty}\frac{2L}{\sqrt{2\pi}}\frac{1}{2L}\int_{-L}^{L} f(x)e^{-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx. \end{aligned}$$

Naturally we have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} dkF(k)e^{ikx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dx' f(x')e^{-ikx'}e^{ikx} \\ &= \int_{-\infty}^{\infty} dx' f(x')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ik(x-x')}\right) = \int_{-\infty}^{\infty} dx' f(x')\delta(x-x'). \end{aligned}$$

Hence we know that

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ik(x-x')}.$$

From this result we can see the Fourier of $f(x)$=1 is $F(k)$=$\sqrt{2\pi}\delta(k)$. The constant function is transformed to the delta function and vice versa.

Example 5.4: Find the Fourier transform of $f(x) = e^{-\alpha^2 x^2}$.

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2 x^2}dx$$

$-\alpha^2x^2 - ikx = -\alpha^2\left(x + \frac{ik}{2\alpha^2}\right)^2 - \frac{k^2}{4\alpha^2}$. Hence we take $u = \alpha(x + ik/2\alpha^2)$

$$F(k) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{k^2}{4\alpha^2}\right)\int_{-\infty+i\frac{k}{2\alpha}}^{\infty+\frac{k}{2\alpha}} e^{-u^2}\frac{du}{\alpha}.$$

Home Work 5.5: Please work out the integral in Example 5.4.

Home Work 5.6: Please find the Fourier Transform of the following functions: (a) $\frac{1}{\cosh ax}$ (b) $e^{-ax^2}\cos bx$ (c) $\frac{x}{x^2+a^2}$.(d) $e^{-\alpha|x|}$, here α is positive.

Parseval's theorem:

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} dx f(x) e^{-ikx} dx,\ G(k) = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} dx g(x) e^{-ikx} dx, \\ &\Longrightarrow \int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F(k)G^*(k)dk. \end{aligned}$$

Proof:

It is actually quite straightforward:

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \infty_{-\infty}^{\infty} dp F(k)G(p)e^{ikx}e^{ipx} \\ &= \int_{-\infty}^{\infty} dk F(k) \infty_{-\infty}^{\infty} dp G(p) \frac{1}{2\pi} dx e^{(k+p)x} \\ &= \int_{-\infty}^{\infty} dk F(k) \infty_{-\infty}^{\infty} dp G(p) \delta(k+p) \\ &= \int_{-\infty}^{\infty} dk F(k) G(-k) \\ &= \int_{-\infty}^{\infty} dk F(k) G^*(k) \end{aligned}$$

Convolution::

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x)e^{-ikx}dx,\ G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx g(x)e^{-ikx}dx, \\ F(k)G(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx h(x)e^{-ikx}dx, h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(u)g(x-u)du. \end{aligned}$$

Proof:

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)G(k)e^{ikx}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk F(k)e^{ikx}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du g(u)e^{-iku} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du g(u)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{ik(x-u)}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du g(u) f(x-u). \end{aligned}$$

Example 5.5 $\frac{\partial\rho}{\partial t}$=$D\frac{\partial^2\rho}{\partial x^2}$ The boundary conditions are $\rho(x,t<0)$=0, $\lim_{x\to\infty}\rho(x,t)\to 0$. $\rho(x,t=0)$=$\frac{m}{A}\delta(x-l)$.

Solution:

First we make Fourier transform of $\rho(x,t)$:

$$\begin{aligned}\tilde{\rho}(k,\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\rho(x,t)e^{-ikx-i\omega t},\\ \rho(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\omega\tilde{\rho}(k,\omega)e^{ikx+i\omega t},\end{aligned}$$

From the equation one has $i\omega$=$-Dk^2$. Therefore $\tilde{\rho}(k,\omega)$=$\tilde{\rho}(k)\delta(\omega+iDk^2)$. So that

$$\rho(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\tilde{\rho}(k)e^{ikx-Dk^2t},$$

The initial condition is $\rho(x,t=0)$=$\frac{m}{A}\delta(x-l)$. To satisfy this condition:

$$\rho(x,t=0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\tilde{\rho}(k)e^{ikx}=\frac{m}{A}\delta(x-l)=\frac{m}{2\pi A}\int_{-\infty}^{\infty}e^{ik(x-l)}dk.$$

Therefore $\tilde{\rho}(k)$=$\frac{m}{A}e^{-ikl}$. Therefore

$$\rho(x,t)=\frac{m}{2\pi A}\int_{-\infty}^{\infty}dke^{-ikl}e^{ikx-Dk^2t}.$$

Example 5.6: Obtain the Green's function satisfying: $\frac{d^2G(t,t')}{dt^2}+2\beta\frac{dG(t,t')}{dt}+\omega_0^2G(t,t')$=$\delta(t-t')$.

Solution:

Making Fourier transform of this equation

$$-\omega^2G(\omega,t')-2i\omega\beta G(\omega,t')+\omega_0^2G(\omega,t')=\frac{1}{\sqrt{2\pi}}e^{i\omega t'}.$$

Therefore

$$G(\omega,t')=\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t'}}{\omega_0^2-2i\beta\omega-\omega^2}.$$

Now we can make $G(t,t')$ by invert the Fourier transform:

$$G(t,t')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega t'}}{\omega_0^2-2i\beta\omega-\omega^2}e^{-i\omega t}.$$

To evaluate this integral we choose the contour C_+=C_1+C_2 for $t<t'$ and C_-=C_1+C_3 for $t>t'$. C_1:[-R,R]. C_2:z=Re^{θ}, $0\le\theta\le\pi$. C_3: z=$Re^{i\theta}$, $\pi\le\theta\le 2\pi$. It is easy to show that

$\int_{C_2} \to 0$ as $R \to \infty$ since

$$
\begin{aligned}
&|\int_{C_2} d\omega \frac{e^{i\omega(t'-t)}}{\omega_0^2 - 2i\beta\omega - \omega^2}| = |\int_0^\pi d\omega \frac{-e^{i\omega(t'-t)}}{\omega^2 + 2i\beta\omega - \omega_0^2}| \\
\leq &\int_0^\pi \frac{e^{-R(t-t')\sin\theta}}{|R^2e^{2i\theta} - 2i\beta R^{i\theta} - \omega_0^2|} R d\theta \\
\leq &\int_0^\pi \frac{e^{-R(t-t')\sin\theta}}{\sqrt{(R^2-\omega_0^2)^2 + 4\beta^2R^2}} R d\theta \\
= &\ 2\int_0^{\pi/2} \frac{e^{-R(t-t')\sin\theta}}{\sqrt{(R^2-\omega_0^2)^2 + 4\beta^2R^2}} R d\theta \\
\leq &\ 2\int_0^{\pi/2} \frac{e^{-2R(t-t')\sin\pi}}{\sqrt{(R^2-\omega_0^2)^2 + 4\beta^2R^2}} R d\theta \\
= &\ \frac{2R}{\sqrt{(R^2-\omega_0)^2 + 4\beta^2R^2}} \frac{1}{2R(t-t')} \left(1 - e^{R(t-t')}\right) \\
\Longrightarrow &\ 0 \ \ R \to \infty
\end{aligned}
$$

Similar for $\int_{C_3}$. Now the poles are $\omega_\pm = -i\beta \pm \sqrt{\omega_0^2 - \beta^2}$. Therefore

$$
\begin{aligned}
&\oint_{C_+} d\omega \frac{e^{i\omega(t'-t)}}{\omega_0^2 - 2i\beta\omega - \omega^2} = 0 \\
&\oint_{C_-} d\omega \frac{e^{i\omega(t'-t)}}{\omega_0^2 - 2i\beta\omega - \omega^2} = -2\pi i(Res(\omega = \omega_+) + Res(\omega = \omega_-)) \\
= &\ -2\pi i \left[\frac{e^{i\omega_+(t'-t)}}{-(\omega_+ - \omega_-)} + \frac{e^{i\omega_-(t'-t)}}{-(\omega_- - \omega_+)}\right] \\
= &\ -2\pi i e^{-\beta(t-t')} \left[\frac{e^{i\sqrt{\omega_0^2-\beta^2}(t'-t)}}{-2\sqrt{\omega_0^2-\beta^2}} + \frac{e^{-i\sqrt{\omega_0^2-\beta^2}(t'-t)}}{2\sqrt{\omega_0^2-\beta^2}}\right] \\
= &\ -2\pi i e^{-\beta(t-t')} \frac{1}{2\sqrt{\omega_0^2-\beta^2}} [-e^{i\sqrt{\omega_0^2-\beta^2}(t'-t)} + e^{-i\sqrt{\omega_0^2-\beta^2}(t'-t)}] \\
= &\ -2\pi i e^{-\beta(t-t')} \frac{1}{2\sqrt{\omega_0^2-\beta^2}} (2i)\sin[\sqrt{\omega_0^2-\beta^2}(t-t')] \\
= &\ 2\pi e^{-\beta(t-t')} \frac{\sin[\sqrt{\omega_0^2-\beta^2}(t-t')]}{\sqrt{\omega_0^2-\beta^2}}
\end{aligned}
$$

Hence we find that

$$
\begin{aligned}
G(t,t') &= 0, \ \ t < t', \\
G(t,t') &= e^{-\beta t} \frac{\sin\sqrt{\omega_0^2-\beta^2}(t-t')}{\sqrt{\omega_0^2-\beta^2}}, \ \ t > t'
\end{aligned}
$$

Home Work 5.7 Please find the Green function satisfying $\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2}$=$\delta(x-x')\delta(t-t')$.
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IV. SINE AND COSINE TRANSFORMATION

Let $f(x)$ is a function only defined for $x \geq 0$. Now let us define a function $\tilde{f}(x)$ as follows:

$$\tilde{f}(x) = f(x), x \geq 0, \tilde{f}(x) = f(-x), x \leq 0.$$

Now let us make the Fourier Transform of $\tilde{f}(x)$:

$$\begin{aligned}
F_c(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(x)e^{-ikx}dx. \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \tilde{f}(x)e^{-ikx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \tilde{f}(x)e^{-ikx}dx. \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(-x)e^{-ikx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)e^{-ikx}dx. \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(y)e^{iky}dy + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)e^{-ikx}dx. \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)(e^{ikx} + e^{-ikx})dx. \\
&= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos kx dx.
\end{aligned}$$

$F_c(x)$ is called the Fourier cosine transform of $f(x)$. It is obvious that $F_c(k)$=$F_c(-k)$. Thus,

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c(k)\cos kx dk = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c(k)\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} F_c(k)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} F_c(k)e^{-ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} F_c(k)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} F_c(-k)e^{-ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} F_c(k)e^{ikx}dk - \frac{1}{\sqrt{2\pi}}\int_{0}^{-\infty} F_c(q)e^{iqx}dq \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_c(k)e^{ikx}dk = \tilde{f}(x).
\end{aligned}$$

Similar we can define $\bar{f}(x)$ as $\bar{f}(x)$=$f(x)$ when $x \geq 0$ but $\bar{f}(x)$=$-f(-x)$ when $x < 0$. Note that the function $\bar{f}(x)$ will be discontinues if $f(x) \neq 0$. Then we have

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(x) e^{-ikx} dx. \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \bar{f}(x) e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \bar{f}(x) e^{-ikx} dx. \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (-f(-x)) e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{-ikx} dx. \\
= & \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (-f(y)) e^{iky} dy + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{-ikx} dx. \\
= & \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)(-e^{ikx} + e^{-ikx}) dx. \\
= & -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin kx dx = -iF_s(k).
\end{aligned}$$

Exercise 5.1 Please prove $\bar{f}(x)$=$\sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx dk$.

Next we have

$$\begin{aligned}
\mathcal{F}_c\left(\frac{df}{dx}\right) &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} dx \frac{df}{dx} \cos kx \\
&= \sqrt{\frac{2}{\pi}} f(x) \cos kx|_{x=0}^{x=\infty} - (-k) \int_{0}^{\infty} f(x) \sin kx dx \\
&= -\sqrt{\frac{2}{\pi}} f(0) + kF_s(k).
\end{aligned}$$

Similarly we have $\mathcal{F}_s\left(\frac{df}{dx}\right)$=$-kF_c(x)$.

Exercise 5.2: Please prove:

(a) $\mathcal{F}_c\left(\frac{d^2f}{dx^2}\right)$=$-\sqrt{\frac{2}{\pi}}\frac{df}{dx}(x=0) - k^2F_c(k)$.

(b) $\mathcal{F}_s\left(\frac{d^2f}{dx^2}\right)$=$k\sqrt{\frac{2}{\pi}}f(x=0) - k^2F_s(k)$.

We can also derive the convolution theorem for the cosine transform:

$$\begin{aligned}\mathcal{F}_c^{-1}[\mathcal{F}_c(f)\mathcal{F}_c(g)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)\mathcal{F}_c(g)\cos kx dk \\ &= \sqrt{\frac{2}{\pi}}\mathcal{F}_c(g)\left(\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\cos k\xi d\xi\right)\cos kx dx \\ &= \frac{2}{\pi}\int_0^\infty dk\int_0^\infty d\xi \mathcal{F}_c(g)f(\xi)\frac{1}{2}[\cos k(x-\xi)+\cos k(x+\xi)] \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi f(\xi)[g(|x-\xi|)+g(x+\xi)]d\xi \\ &.\end{aligned}$$

Naturally we have similar theorem for the sine transform:

$$\begin{aligned}\mathcal{F}_s^{-1}[\mathcal{F}_s(f)\mathcal{F}_s(g)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)\mathcal{F}_c(g)\sin kx dk \\ &= \sqrt{\frac{2}{\pi}}\mathcal{F}_c(g)\left(\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\sin k\xi d\xi\right)\cos kx dx \\ &= \frac{2}{\pi}\int_0^\infty dk\int_0^\infty d\xi \mathcal{F}_c(g)f(\xi)\frac{1}{2}[\cos k(x-\xi)-\cos k(x+\xi)] \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi f(\xi)[\tilde{g}(|x-\xi|)+\tilde{g}(x+\xi)]d\xi \\ &.\end{aligned}$$

Here $\tilde{g}$=$\mathcal{F}_c^{-1}[\mathcal{F}_s(g)]$. The following is an example of applying sine transform to solve differential equation:

Example 5.7 Use a Fourier sine transform to solve the equation $\frac{d^2y}{dx^2} - a^2y$=$f(x)$ with the boundary conditions $y(x=0)$=y_0 and $\lim_{x\to\infty} y(x)$=0.

Solution:

Taking the sine transformations of the both side of the equation:

$$-k^2\mathcal{F}_s(y) + k\sqrt{\frac{2}{\pi}}y_0 - a^2\mathcal{F}_s(y) = \mathcal{F}_s(f).$$

Naturally we have

$$\mathcal{F}_s(y) = \sqrt{\frac{2}{\pi}}\frac{ky_0}{k^2+a^2} - \frac{1}{k^2+a^2}\mathcal{F}_s(f).$$

Hence one has

$$y(x) = y_0\sqrt{\frac{2}{\pi}}\mathcal{F}_s^{-1}\left(\frac{k}{k^2+a^2}\right) - \mathcal{F}_s^{-1}\left(\frac{1}{k^2+a^2}\mathcal{F}_s(f)\right).$$

The first term is obtained by

$$y_0\sqrt{\frac{2}{\pi}}\mathcal{F}_s^{-1}\left(\frac{k}{k^2+a^2}\right)$$
$$= \frac{2y_0}{\pi}\int_0^\infty \frac{k}{k^2+a^2}\sin kx dk = y_0 e^{-ax}.$$

The second term is more complicated. We need apply the convolution theorem derived before. One needs calculate the following quantity first:

$$\mathcal{F}_c^{-1}\left(\frac{1}{k^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos kx}{k^2+a^2}dk = \sqrt{\frac{\pi}{2}}\frac{e^{-ax}}{a}.(x>0)$$

Therefore one has

$$\mathcal{F}_s^{-1}\left(\frac{1}{k^2+a^2}\mathcal{F}_s(f)\right) = \frac{1}{2a}\int_0^\infty f(\xi)(e^{-a|x-\xi|}-e^{-a|x+\xi|})d\xi$$
$$= \frac{1}{2a}\left(\int_0^x f(\xi)e^{-a(x-\xi)}d\xi + \int_x^\infty f(\xi)e^{-a(\xi-x)}d\xi - \int_0^\infty e^{-a(x+\xi)})d\xi\right)$$
$$= \frac{e^{-ax}}{2a}\left(\int_0^x f(\xi)e^{a\xi}d\xi - \int_0^\infty e^{-a\xi}d\xi\right) + \frac{e^{ax}}{2a}\int_x^\infty f(\xi)e^{-a\xi}d\xi$$

At end we obtain

$$y(x) = y_0 e^{-ax} + \frac{e^{-ax}}{2a}\left(\int_0^x f(\xi)e^{a\xi}d\xi - \int_0^\infty e^{-a\xi}d\xi\right) + \frac{e^{ax}}{2a}\int_x^\infty f(\xi)e^{-a\xi}d\xi.$$

Note that when $x\to\infty$, $y\to 0$.

Home Work 5.8:Please find the solution of $\frac{\partial T}{\partial t}{=}D\frac{\partial^2 T}{\partial x^2}$ with boundary conditions $T(x=0,t){=}f(t)$ and $T(x,t=0){=}0$ for $x>0$. Here $T(x,t)$ is only defined at $x\geq 0$.