Lecture VI: Laplace Transformation

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

(Dated: August 25, 2012)

This lecture introduce Laplace transformation.

I. BASIC PROPERTIES OF LAPLACE TRANSFORMATION

Besides Fourier transformation, there are other similar integral transformations. Among them the most often used one is Laplace transformation. The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}(f) = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Here exists a real positive constant M such that $|e^{-\sigma_0 t}f(t)| \le M$ for all $0 < t < \infty$. σ_0 is real. The transform is defined for $Re(s) \ge \sigma_0$. Let us first to see some simple examples:

Example 6.1:Find the Laplace transforms of (a) $f(t)$=$e^{\alpha t}$ (b) $f(t)$=t^p (c)$f(t)$=$\cos\omega t$.

Solution:

(a):

$$F(s) = \int_0^\infty e^{\alpha t}e^{-st}dt = \frac{1}{\alpha - s}e^{(\alpha-s)t}|_{t=0}^{t\infty} = \frac{1}{s-\alpha}.$$

(b):

$$F(s) = \int_0^\infty t^p e^{-st}dt = \int_0^\infty \left(\frac{u}{s}\right)^p e^{-u}\frac{du}{s} = \frac{1}{s^{p+1}}\int_0^\infty u^p e^{-u}du = \frac{\Gamma(p+1)}{s^{p+1}}.$$

(c):

$$F(s) = \int_0^\infty \frac{e^{i\omega t}+e^{-i\omega t}}{2}e^{-st}dt = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) = \frac{s}{s^2+\omega^2}.$$

Example 6.2: Prove that (a) $\mathcal{L}(\frac{df}{dt}) = -f(0) + sF(s)$ (b) $\mathcal{L}(e^{-at}f) = F(s+a)$ (c) $\int_s^\infty F(\sigma)d\sigma = \mathcal{L}\left(\frac{f}{t}\right)$.

Solution:

(a):

$$\int_0^\infty \frac{df}{dt}(t)e^{-st}dt = f(t)e^{-st}|_{t=0}^{t=\infty} - \int_0^\infty (-s)f(t)e^{-st}dt = -f(0) + sF(s).$$

(b):

$$\int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = F(s+a).$$

(c):

$$\int_s^\infty F(\sigma)d\sigma = \int_s^\infty d\sigma \int_0^\infty dt f(t)e^{-\sigma t} = \int_0^\infty dt f(t)\frac{e^{-\sigma t}}{-t}|_s^\infty = \int_0^\infty \frac{f(t)}{t}e^{-st}dt.$$

Example 6:3:(a) $\Theta(t-t_0)$=1 when $t > t_0$; $\Theta(t-t_0)$=0 when $t < t_0$ then $\mathcal{L}(\Theta(t-t_0)f(t-t'))$=$e^{-st_0}F(s)$.

(b)If $f(t)$ is periodic function with period T and $f(t+nT)$=$g(t)$ as $nT \leq t \leq (n+1)T$. $g(t)$ is zero except in $[0,t]$. Please show that $F(s)$=$\frac{G(s)}{1-e^{sT}}$,$G(s)$=$\int_0^T dtg(t)e^{-st}dt$.

Solution:

(a):

$$\mathcal{L}(\Theta(t-t_0)f(t-t_0)) = \int_0^\infty \Theta(t-t_0)f(t-t_0)e^{-st}dt = \int_{t_0}^\infty f(t-t_0)e^{-st}dt$$
$$= \int_0^\infty f(u)e^{-s(u+t_0)}du = e^{-st_0}\int_0^\infty f(u)e^{-su}du.$$

(b):

$$f(t) = g(t) + \Theta(t-T)g(t-T) + \theta(t-2T)g(t-2T) + ... = \sum_{n=0}^{\infty} S(t-nT)g(t-nT).$$

Since we have $\mathcal{L}(\Theta(t-t_0)f9t-t_0))$=$e^{-st_0}F(s)$. So that

$$F(s) = (1 + e^{-sT} + e^{-2sT} + ...)G(s) = \frac{G(s)}{1-e^{-sT}}.$$

Home Work 6.1: Prove that (a) $\mathcal{L}(tf) = -\frac{dF}{ds}(s)$ (b) $\mathcal{L}(\frac{1-\cos t}{t}) = \frac{1}{2}\ln\left(1+\frac{\omega^2}{s^2}\right)$.

Home Work 6.2: Find (a) $\mathcal{L}(te^{\sqrt{t}})$ (b) $\mathcal{L}\left(\frac{\sinh \alpha t}{t}\right)$.

Home Work 6.3 Please find the inverse Laplace transform of (a) $\frac{s}{s^2+2s+3}$ (b)$\frac{1}{(s^2+a^2)^2}$.

II. APPLICATION OF LAPLACE TRANSFORMATION

Example 6.4: $\frac{d^4y}{dx^4}$=$q(x)$, here $q(x) = K, (L-l)/2 \leq x \leq (L+l)/2$, $q(x)$=0 otherwise. Here boundary conditions are $y(0)$=$y(L)$=0. $y''(0)$=$y''(L)$=0.

Solution:

Making Laplace transform of the equation:

$$s^4Y(s) - s^3y(0) - s^2\frac{dy}{dx}(x=0) - s\frac{d^2y}{dx^2}(x=0) - \frac{d^3y}{dx^3}(x=0) = Q(s).$$

$$Q(s) = \int_{(L-l)/2}^{(L+l)/2} Ke^{-sx}dx = \frac{-K}{s}\left[\exp\left(-\frac{L+l}{2}s\right) - \exp\left(-\frac{L-l}{2}s\right)\right] = \frac{2K}{s}e^{-Ls/2}\sinh\frac{ls}{2}.$$

$$Y(s) = \frac{2K}{s^5}e^{-Ls/2}\sinh\frac{ls}{2} + \frac{1}{s^2}\frac{dy}{dx}(x=0) + \frac{1}{s^4}\frac{d^3y}{dx^3}(x=0).$$

We have

$$\begin{aligned} y(x) &= 2K\mathcal{L}^{-1}(\frac{1}{s^5}e^{-Ls/2}\sinh\frac{ls}{2}) + \frac{dy}{dx}(x=0)\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \frac{d^3y}{dx^3}(x=0)\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) \\ &= \frac{K}{24}\left(x-\frac{L+l}{2}\right)^4\Theta\left(x-\frac{L+l}{2}\right) - \frac{K}{24}\left(x-\frac{L+l}{2}\right)^4\Theta\left(x-\frac{L+l}{2}\right) \\ &+ x\frac{dy}{dx}(x=0) + \frac{x^3}{6}\frac{d^3y}{dx^3}(x=0). \end{aligned}$$

Now we determine the value of $\frac{dy}{dx}(x=0)$ and $\frac{d^3y}{dx^3}(x=0)$.

$$\begin{aligned} y(x=L) &= \frac{K}{24}\left(L-\frac{L+l}{2}\right)^4 - \frac{K}{24}\left(L-\frac{L+l}{2}\right)^4 \\ &+ L\frac{dy}{dx}(x=0) + \frac{L^3}{6}\frac{d^3y}{dx^3}(x=0). \\ &= \frac{K}{24}\frac{1}{2}Ll(L^2+l^2) + L\frac{dy}{dx}(x=0) + \frac{L^3}{6}\frac{d^3y}{dx^3}(x=0) = 0. \end{aligned}$$

$$\frac{d^3y}{dx^3}(x=L) = \frac{K}{2}\left(L-\frac{L+l}{2}\right)^2 - \frac{K}{2}\left(L-\frac{L+l}{2}\right)^2 + L\frac{d^3y}{dx^3}(x=0) = 0.$$

Hence we determine the values of $\frac{dy}{dx}(x=0)$ and $\frac{d^3y}{dx^3}(x=0)$ as

$$\frac{d^3y}{dx^3}(x=0) = -\frac{Kl}{2},\ \frac{dy}{dx}(x=0) = \frac{Kl}{48}(3L^2-l^2).$$

Therefore we obtain the answer

$$\begin{aligned} y(x) &= \frac{K}{24}\left(x-\frac{L+l}{2}\right)^4\Theta\left(x-\frac{L+l}{2}\right) - \frac{K}{24}\left(x-\frac{L+l}{2}\right)^4\Theta\left(x-\frac{L+l}{2}\right) \\ &+ Kl\left(\frac{3L^2-l^2}{2}\right)x - 2Klx^3. \end{aligned}$$

Home Work 6.4 Use Laplace transform to solve the equation $t\frac{dy}{dt} + y=e^{-t}$.

Home Work 6.5 Use Laplace transform to solve the equation :$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y=0$.

III. CONVOLUTION

:

$$\mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) = F(s)G(s).$$

Proof:

$$\begin{aligned}
&\mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) = \lim_{T\to\infty}\int_0^T dt\int_0^t d\tau e^{-st}f(\tau)g(t-\tau)\\
&= \lim_{T\to\infty}\int_0^T d\tau\int_\tau^T dte^{-st}f(\tau)g(t-\tau) = \lim_{T\to\infty}\int_0^T d\tau\int_0^{T-\tau} due^{-s(u+\tau)}f(\tau)g(u)\\
&= \lim_{T\to\infty}\int_0^T d\tau\left(\int_0^{T-\tau} due^{-su}g(u)\right)f(\tau)e^{-s\tau}\\
&= \lim_{T\to\infty}\left(\int_0^{T/2}d\tau\int_0^{T/2}du + \int_{T/2}^T d\tau\int_0^{T-\tau}du + \int_0^{T/2}d\tau\int_{T/2}^{T-\tau}du\right)e^{-su}g(u)f(\tau)e^{-s\tau}\\
&= I + II + III.
\end{aligned}$$

Next we can show II and III both converge to zero when T approaches infinity. Because $|g(u)| \le M_1 e^{\sigma_1 u}$ and $|f(\tau)| \le M_2 e^{\sigma_2 \tau}$ for $0 < u < T/2$ $T/2 < \tau < T$. Therefore

$$\begin{aligned}
&|II| \le M_1M_2|\int_{T/2}^T e^{(\sigma_2-s)\tau}d\tau\int_0^{T-\tau} e^{(\sigma_1-u)}du|\\
&= M_1M_2|\int_{T/2}^T e^{(\sigma_2-s)\tau}\frac{1-e^{(\sigma_1-s)(T-\tau)}}{s-\sigma_1}d\tau|\\
&= \frac{M_1M_2}{s-\sigma_1}\left[\frac{e^{(\sigma_2-s)T/2}-e^{(\sigma_2-s)T}}{s-\sigma_2} - e^{-sT}\frac{e^{\sigma_2 T}-e^{(\sigma_1+\sigma_2)T/2}}{\sigma_2-\sigma_1}\right] \to 0 as T \to \infty.
\end{aligned}$$

Similarly III also approaches zero when T approaches infinity. Therefore we only need evaluate I

$$\lim_{T\to\infty}\int_0^{T/2}d\tau\left(\int_0^{T/2}due^{-su}g(u)f(\tau)e^{-s\tau}\right) = \int_0^\infty e^{-su}g(u)du\int_0^\infty e^{-s\tau}f(\tau)du = F(s)G(s).$$

Home Work 6.6 Please find the inverse Laplace transform of $\frac{\omega s}{s^4-\omega^4}$.

Example:Please solve $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}_0\cos\Omega t$ by Laplace transform.

Solution:

First we make Laplace transform of the first term of LHS: $\mathcal{L}\left(\frac{di}{dt}\right)$=$\mathcal{L}\left(\frac{d^2q}{dt^2}\right)$=$s^2Q(s)-\frac{dq}{dt}(t=0)-sq(t=0)$. The second term is also transformed as $\mathcal{L}\left(\frac{dq}{dt}\right)$=$-q(t=0)+sQ(s)$. Hence we make Laplace transform of both sides of the equation:

$$s^2LQ(s)+sRQ(s)+\frac{Q(s)}{C}=E(s)\Longrightarrow Q(s)=\frac{E(s)}{s^2L+sR+\frac{1}{C}}=E(s)R(s).$$

$$R(s)=\frac{1}{Ls^2+sR+\frac{1}{C}}=\frac{1}{L}\left(\frac{1}{(s+\alpha)^2+\omega^2}\right).$$

α=$\frac{R}{2L}$ and ω^2=$\frac{1}{LC}-\frac{R^2}{L^2}$=$\omega_0^2-\alpha^2$. The inverse Laplace transform of $R(s)$ is $r(t)$=$\frac{1}{L\omega}e^{-\alpha t}\sin\omega t$. Hence

$$q(t)=\int_0^t d\tau\frac{\mathcal{E}_0}{L\omega}\cos\Omega\tau\sin\omega(t-\tau)\exp[-\alpha(t-\tau)].$$

IV. MELLIN TRANSFORMATION

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)e^{st}ds.$$

$$\begin{aligned}\mathcal{L}(f(t)) &= \int_0^\infty dte^{-st}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(\sigma)e^{\sigma t}d\sigma\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\sigma F(\sigma)\int_0^\infty dte^{(\sigma-s)t}\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\sigma F(\sigma)\frac{e^{-(s-\sigma)t}}{\sigma-s}|_0^\infty\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\sigma\frac{F(\sigma)}{s-\sigma}\end{aligned}$$

$$\mathcal{L}(f(t))=-2\pi iRes(\sigma=s)=F(s).$$

Example: Find the function $f(t)$ such that $\mathcal{L}(f(t))$=$\frac{1}{\sqrt{s}}$.

Solution:

Choose the contour C=C_1+C_2+C_3+C_4+C_5+C_6. C_1:$[\gamma-iR,\gamma+iR]$. C_2:$z=\gamma+Re^{i\theta},\pi/2\leq\theta\leq\pi-\epsilon'$. C_3:$[\gamma-R+i\rho\sin\epsilon,-\rho\cos\epsilon+i\rho\sin\epsilon]$. C_4: $z=\rho e^{i\theta}$,θ runs from $\pi-\epsilon$ to $\pi+\epsilon$ clockwise. C_5:$[-\rho\cos\epsilon-i\rho\sin\epsilon,\gamma-R-i\rho\sin\epsilon]$, C_6: $z=\gamma+Re^{itheta}$,$\pi+\epsilon'\leq\theta\leq 3\pi/2$

$$\oint_C\frac{e^{st}}{\sqrt{s}}ds=0.$$

First we show that $\int_{C_2}$ and $\int_{C_4}$ approaches zero when $R \to \infty$.

$$
\begin{aligned}
&|\int_{C_2} \frac{e^{st}}{\sqrt{s}} ds| = |\int_{\pi/2}^{\pi-\epsilon'} \frac{e^{\gamma t} e^{Rt\cos\theta} e^{iRt\sin\theta}}{\sqrt{\gamma + Re^{i\theta}}} Re^{i\theta} i d\theta| \\
&\leq \int_{\pi/2}^{\pi-\epsilon'} \frac{|e^{\gamma t} e^{Rt\cos\theta} e^{iRt\sin\theta}|}{|\sqrt{\gamma + Re^{i\theta}}|} |Re^{i\theta} i| d\theta = \int_{\pi/2}^{\pi-\epsilon'} \frac{|e^{\gamma t} e^{Rt\cos\theta}|}{|\sqrt{\gamma + Re^{i\theta}}|} R d\theta \\
&\leq e^{\gamma t} \int_{\pi/2}^{\pi-\epsilon'} \frac{e^{Rt\cos\theta}}{|\sqrt{R-\gamma}|} R d\theta = e^{\gamma t} \int_{0}^{\pi/2-\epsilon'} \frac{e^{-Rt\sin\xi}}{|\sqrt{R-\gamma}|} R d\xi \\
&\leq e^{\gamma t} \int_{0}^{\pi/2-\epsilon'} \frac{e^{-2Rt\xi/\pi}}{\sqrt{R-\gamma}} R d\xi \leq e^{\gamma t} \frac{\pi R}{2R\sqrt{R-\gamma}} \left(1 - e^{-Rt}\right) \to 0.
\end{aligned}
$$

Here we have $\xi = \theta - \frac{\pi}{2}$.

$$
\begin{aligned}
&|\int_{C_6} \frac{e^{st}}{\sqrt{s}} ds| = |\int_{\pi+\epsilon'}^{3\pi/2} \frac{e^{\gamma t} e^{Rt\cos\theta} e^{iRt\sin\theta}}{\sqrt{\gamma + Re^{i\theta}}} Re^{i\theta} i d\theta| \\
&\leq \int_{\pi+\epsilon'}^{3\pi/2} \frac{|e^{\gamma t} e^{Rt\cos\theta} e^{iRt\sin\theta}|}{|\sqrt{\gamma + Re^{i\theta}}|} |Re^{i\theta} i| d\theta = \int_{3\pi/2}^{\pi+\epsilon'} \frac{|e^{\gamma t} e^{Rt\cos\theta}|}{|\sqrt{\gamma + Re^{i\theta}}|} R d\theta \\
&\leq e^{\gamma t} \int_{\pi+\epsilon'}^{3\pi/2} \frac{e^{Rt\cos\theta}}{|\sqrt{R-\gamma}|} R d\theta = e^{\gamma t} \int_{\pi/2-\epsilon'}^{0} \frac{e^{-Rt\sin\xi}}{|\sqrt{R-\gamma}|} R(-d\xi) \\
&\leq e^{\gamma t} \int_{0}^{\pi/2-\epsilon'} \frac{e^{-2Rt\xi/\pi}}{\sqrt{R-\gamma}} R d\xi = e^{\gamma t} \frac{\pi R}{2R\sqrt{R-\gamma}} \left(1 - e^{-Rt}\right) \to 0.
\end{aligned}
$$

Here we have $\xi = 3\pi/2 - \theta$. Next we have

$$
\begin{aligned}
&|\int_{C_4} \frac{e^{st}}{\sqrt{s}} ds| = |\int_{\pi-\epsilon'}^{-\pi/2+\epsilon'} \frac{e^{\gamma t} e^{\rho t\cos\theta} e^{i\rho t\sin\theta}}{\sqrt{\gamma + \rho e^{i\theta}}} Re^{i\theta} i d\theta| \\
&\leq \int_{-\pi/2+\epsilon'}^{\pi-\epsilon'} \frac{|e^{\gamma t} e^{\rho t\cos\theta} e^{i\rho t\sin\theta}|}{|\sqrt{\gamma + \rho e^{i\theta}}|} |\rho e^{i\theta} i| d\theta = \int_{-\pi+\epsilon'}^{\pi-\epsilon'} \frac{|e^{\gamma t} e^{\rho t\cos\theta}|}{|\sqrt{\gamma + \rho e^{i\theta}}|} \rho d\theta \\
&\leq e^{\gamma t} \int_{-\pi+\epsilon'}^{\pi-\epsilon'} \frac{e^{\rho t\cos\theta}}{|\sqrt{\rho-\gamma}|} \rho d\theta \leq e^{\gamma t} 2\pi \frac{\rho}{\sqrt{\gamma}} \to 0
\end{aligned}
$$

as $\rho \to 0$. Therefore under the limits $R \to \infty, \rho \to 0$ we have

$$
\int_{C_1} \frac{e^{st}}{\sqrt{s}} ds = -\left(\int_{C_3} + \int_{C_5}\right) \frac{e^{st}}{\sqrt{s}} ds.
$$

$$
\int_{C_3} \frac{e^{st}}{\sqrt{s}} ds = \int_{0}^{\infty} \frac{e^{-rt}}{\sqrt{re^{i\pi}}} dr = \frac{1}{i} \int_{0}^{\infty} \frac{e^{-rt}}{\sqrt{r}} dr.
$$

$$
\int_{C_5} \frac{e^{st}}{\sqrt{s}} ds = \int_{\infty}^{0} \frac{e^{-rt}}{\sqrt{re^{-i\pi}}} dr = \frac{1}{i} \int_{0}^{\infty} \frac{e^{-rt}}{\sqrt{r}} dr.
$$

Hence we obtain

$$
\int_{\gamma-i\infty}^{\gamma+i\infty} = -1 \cdot 2 \int_{0}^{\infty} \frac{e^{-rt}}{i\sqrt{r}} dr = 2i \int_{0}^{\infty} 2e^{-rt} d\sqrt{r} = 2i\sqrt{\frac{\pi}{t}} = 2i\sqrt{\frac{\pi}{t}}.
$$

Hence

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+\infty}\frac{e^{st}}{\sqrt{s}}ds=\frac{1}{\sqrt{\pi t}}.$$

Home Work 6.7 Using the Mellin inversion integral with branch cut from $-a$ to a along the real axis to obtain the inverse Laplace transfrom of $F(s)=\frac{1}{\sqrt{s^2-a^2}}$. (Hint: change variable to z, $s=\frac{a}{2}\left(z+\frac{1}{z}\right)$.)

Home Work 6.8: Show by evaluating a Mellin inversion integral that
$\mathcal{L}^{-1}\left(\frac{s}{e}-\sqrt{s/kx}\right)=erfc\left(\frac{x}{2\sqrt{kt}}\right)$. Here $erfc(y)=\frac{2}{\sqrt{\pi}}\int_y^{\infty}e^{-u^2}du$.