# Lecture VII: Speical Function: Legendre Functions

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

(Dated: August 18, 2012)

This lecture introduce Legendre functions

## I. GENERATING FUNCTION OF LEGENDRE POLYNOMIALS

The solution for Poisson equation

$$\nabla^2 u(r,\theta,\phi) = \rho(r,\theta) = -4\pi\delta(x)\delta(y)\delta(z-r_s).$$

can be obtained from the result of Chapter 1. It is

$$\begin{aligned} u(r,\theta,\phi) &= \int d^3r' G(\vec{r},\vec{r'})\rho(\vec{r'}) = \int d^3r' \frac{\delta(x')\delta(y')\delta(z'-r_s)}{|\vec{r}-\vec{r'}|} = \frac{1}{|\vec{r}-r_s\hat{e}_z|} \\ &= \frac{1}{(r^2-2rr_s\cos\theta+r_s^2)^{1/2}} = \frac{1}{r_s}\frac{1}{\left(1-2\frac{r}{r_s}\cos\theta+\left(\frac{r}{r_s}\right)^2\right)^{1/2}}. \end{aligned}$$

The above solution is only for $r < r_s$. Now if we set $r_s$=1 we can expand the solution as

$$u(r,\theta) = \sum_{l=0}^{\infty} r^l P_l(\cos\theta) = (1-2r\cos\theta+r^2)^{-1/2}.$$

It can be written as

$$\frac{1}{\sqrt{1-2x\mu+x^2}} = \sum_{l=0}^{\infty} x^l P_l(\mu).$$

On the other hand, if we limit ourselves to $r < 1$ then the Poisson equation is just like Laplace equation. In spherical coordinate the solution $u(r,\theta)$=$R(r)P(\theta)$ which is independent of $\phi$ should satisfy those equations:

$$\begin{aligned} &\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = kR. \\ &\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + kP = 0. \end{aligned}$$

It is easy to see if $P(\mu)$ is polynomial, the value of $k$ must be $l(l+1)$. Here $l$ is an integer. Hence we have

$$R(r) = \sum_{l=0}^{\infty} A_l r^l + \frac{B_l}{r^{l+1}}.$$

Since we consider the solution for $0 < r < 1$. To match the two solution we know $P_l(\mu)$ must satsify the equation:

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_l}{d\mu}\right) + l(l+1)P_l = 0.$$

**Home Work 7.1**: Please prove that if $P(\mu)$ is polynomial, the value of $k$ must be $l(l+1)$.

## II. LEGENDRE FUNCTIONS

$$\frac{1}{\sqrt{1-2x\mu+x^2}} = \sum_{l=0}^{\infty} x^l P_l(\mu).$$

Since we have

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n-1)!}{2^{2n} n!(n-1)!} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^n.$$

$$\begin{aligned}
&(1-2x\mu+x^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(-2x\mu+x^2)^n \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(-2\mu)^k x^{2n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(2n)!}{2^{2n-k} n!(n-k)!k!}(-1)^k x^{2n-k}\mu^k \\
&= \sum_{n=0}^{\infty}\sum_{l=n}^{2n} \frac{(2n)!}{2^l n!(l-n)!(2n-l)!}(-1)^{2n-l} x^l \mu^{2n-l} = \sum_{l=0}^{\infty}\sum_{n=[l/2]}^{l} \frac{(2n)!}{2^l n!(l-n)!(2n-l)!}(-1)^{2n-l} x^l \mu^{2n-l} \\
&= \sum_{l=0}^{\infty}\sum_{r=0}^{l/2} \frac{(2l-2r)!}{2^l (l-r)!r!(l-2r)!}(-1)^{l-2r} x^l \mu^{l-2r} = \sum_{l=0}^{\infty}\sum_{r=0}^{l/2} \frac{(2l-2r)!}{2^l (l-r)!r!(l-2r)!}(-1)^{l} x^l \mu^{l-2r} \\
&= \sum_{l=0}^{\infty} x^l P_l(\mu)
\end{aligned}$$

$$P_l(\mu) = \sum_{r=0}^{l/2} \frac{(2l-2r)!}{2^l (l-r)!r!(l-2r)!}(-1)^l x^l \mu^{l-2r}$$

**Rodrigues Formula**:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

**Proof:**

$$\begin{aligned}
(x^2-1)^l &= \sum_{r=0}^{l} \frac{l!}{r!(l-r)!}(-1)^l x^{2l-2r}. \\
\frac{d^l}{dx^l}(x^2-1)^l &= \sum_{r=0}^{l} \frac{l!}{r!(l-r)!}(-1)^l \frac{d^l}{dx^l} x^{2l-2r} = \sum_{r=0}^{l/2} \frac{l!}{r!(l-r)!}(-1)^l \frac{(2l-2r)!}{(l-2r)!} x^{l-2r} \\
\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l &= \sum_{r=0}^{l/2} \frac{l!}{2^l l! r!(l-r)!}\frac{(2l-2r)!}{(l-2r)!}(-1)^l x^{l-2r} = \sum_{r=0}^{l/2} \frac{(2l-2r)!}{2^l r!(l-r)!(l-2r)!}(-1)^l x^{l-2r}
\end{aligned}$$

**Orthogonality**

$$\begin{aligned}
&\int_{-1}^{+1} \frac{1}{1-2x\mu+\mu^2} d\mu = \int_{-1}^{+1} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x^k x^l P_k(\mu) P_l(\mu) d\mu \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x^{k+l} \int_{-1}^{+1} P_k(\mu) P_l(\mu) d\mu \\
&= \frac{-1}{2x}\int_{(1+x)^2}^{(1-x)^2} \frac{dv}{v} = \frac{1}{x}\ln\frac{1+x}{1-x} \\
&= 2\left(1+\frac{x^2}{3}+...+\frac{x^{2l}}{2l+1}+...\right) = \sum_{l=0}^{\infty} x^{2l} \int_{-1}^{+1} P_l(\mu) P_l(\mu) d\mu
\end{aligned}$$

Hence

$$\int_{-1}^{+1} P_k(\mu) P_l(\mu) d\mu = \frac{2}{2l+1}\delta_{lk}.$$

**Recursion Relation**:

$$\begin{aligned}
&\frac{\partial G}{\partial x} = -\frac{x-\mu}{(1-2x\mu+\mu^2)^{3/2}} = -\frac{x-\mu}{1-2x\mu+\mu^2}G. \\
\Longrightarrow\ &(1-2x\mu+\mu^2)\frac{\partial G}{\partial x} = (\mu-x)G. \\
&(1-2x\mu+\mu^2)\sum_{l=0}^{\infty} l x^{l-1}P_l(\mu) = (\mu-x)\sum_{l=0}^{\infty} x^l P_l(\mu). \\
&\sum_{l=0}^{\infty}(l+1)x^{l+1}P_l(\mu) - \mu\sum_{l=0}^{\infty}(2l+1)x^l P_l(\mu) + \sum_{l=0}^{\infty} l x^{l-1}P_l(\mu) = 0. \\
&lP_{l-1}(\mu) - (2l+1)\mu P_l(\mu) + (l+1)P_{l+1}(\mu) = 0.
\end{aligned}$$

**Home Work 7.2**: Please prove that $lP_l(\mu)=\mu P_l'(\mu) - P_{l-1}'(\mu)$. (Hint: Using $\frac{\partial G}{\partial \mu}$).

**Home Work 7.3**: Please use the previous results to prove that

(a) $(2l+1)P_l(\mu)=P_{l+1}'(\mu) - P_{l-1}'(\mu)$

(b) $P_{l-1}(\mu)=\mu P_l(\mu) + \left(\frac{1-\mu^2}{l}\right) P_l'(\mu)$

(c) $P_{l+1}(\mu)=\mu P_l(\mu) - \left(\frac{1-\mu^2}{l+1}\right) P_l'(\mu)$

**Example 7.1**: $\nabla^2 V(r,\theta,\phi)=0$ with $V=V_0$ for $r=a$, $0\le\theta\le\frac{\pi}{2}$. $V=0$ for $r=a$, $\frac{\pi}{2}\le\theta\le\pi$. Please find the solution for $r\le a$.

Solution:

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

$$\begin{aligned}
V(r=a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\theta) &= V_0, 0\le\cos\theta\le 1, \\
&= 0, -1\le\cos\theta\le 0.
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{+1}\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)P_k(\cos\theta)d\cos\theta &= \int_0^{+1} V_0 P_k(\cos\theta)d\cos\theta. \\
\Longrightarrow A_l a^l\frac{2}{2l+1} &= V_0\int_0^{+1} P_l(\cos\theta)d\cos\theta.
\end{aligned}$$

**Home Work 7.3** Please show that $\int_0^{+1} P_l(\cos\theta) d\cos\theta {=} -\frac{P_{l+1}(0)}{l}$. $l \geq 1$.

From the above result we have

$$A_l = -\frac{V_0}{2}\frac{2l+1}{la^l}P_{l+1}(0),\ \ l \neq 0.$$

For $l{=}0$ we have $A_0{=}\frac{V_0}{2}$. Therefore we obtain $V(r,\theta\phi){=}\frac{V_0}{2}\left[1-\sum_{l=1}^{\infty}\frac{2l+1}{l}P_{l+1}(0)\left(\frac{r}{a}\right)^l P_l(\cos\theta)\right]$.

## III. ASSOCIATED LEGENDRE FUNCTIONS

If we go beyond the axial symmetric situation, the Laplace equation in spherical coordinate become as follows:

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) - \frac{m^2}{1-\mu^2}P + l(l+1)P = 0.$$

WE find the solutions are related by Legendre polynomials in an interesting way. Remember that

$$(1-\mu^2)\frac{d^2P_l}{d\mu^2} - 2\mu\frac{dP_l}{d\mu} + l(l+1)P_l = 0.$$

Taking m times derivatives one has the following identity:

$$\frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{d^2P_l}{d\mu^2}\right] - \frac{d^m}{d\mu^m}\left[2\mu\frac{dP_l}{d\mu}\right] + l(l+1)\frac{d^mP_l}{d\mu^m} = 0$$

$$\frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{d^2P_l}{d\mu^2}\right] = (1-\mu^2)\frac{d^{m+2}P_l}{d\mu^{m+2}} - 2m\mu\frac{d^{m+1}P_l}{d\mu^{m+1}} - \frac{m(m-1)}{2}2\frac{d^mP_l}{d\mu^m}.$$

$$\frac{d^m}{d\mu^m}\left[2\mu\frac{dP_l}{d\mu}\right] = 2\mu\frac{d^{m+1}P_l}{d\mu^{m+1}} + 2m\frac{d^mP_l}{d\mu^m}.$$

$$\Longrightarrow (1-\mu^2)\frac{d^{m+2}P_l}{d\mu^{m+2}} - 2m\mu\frac{d^{m+1}P_l}{d\mu^{m+1}} - m(m-1)\frac{d^mP_l}{d\mu^m} - 2\mu\frac{d^{m+1}P_l}{d\mu^{m+1}} - 2m\frac{d^mP_l}{d\mu^m} + l(l+1)\frac{d^mP_l}{d\mu^m} = 0$$

$$(1-\mu^2)\frac{d^{m+2}P_l}{d\mu^{m+2}} - 2(m+1)\mu\frac{d^{m+1}P_l}{d\mu^{m+1}} + [l(l+1)-m(m+1)]\frac{d^mP_l}{d\mu^m} = 0.$$

Set $F(\mu)$=$(1-\mu^2)^{m/2}\frac{d^m P_l}{d\mu^m}$ then

$$
\begin{aligned}
& (1-\mu^2)\frac{d^2}{d\mu^2}\left[(1-\mu^2)^{-m/2}F(\mu)\right] - 2(m+1)\mu\frac{d}{d\mu}\left[(1-\mu^2)^{-m/2}F(\mu)\right] \\
+ \quad & [l(l+1)-m(m+1)](1-\mu^2)^{-m/2}F(\mu) = 0 \\
& \frac{d}{d\mu}\left[(1-\mu^2)^{-m/2}F(\mu)\right] = (1-\mu^2)^{-m/2}\frac{dF(\mu)}{d\mu} + m\mu(1-\mu^2)^{-m/2-1}F(\mu), \\
& \frac{d^2}{d\mu^2}\left[(1-\mu^2)^{-m/2}F(\mu)\right] = (1-\mu^2)^{-m/2}\frac{d^2F(\mu)}{d\mu^2} + 2m(1-\mu^2)^{-m/2-1}\mu\frac{dF(\mu)}{d\mu} \\
+ \quad & m(m+2)(1-\mu^2)^{-m/2-2}\mu^2F(\mu), \\
\Longrightarrow \quad & (1-\mu^2)^{-m/2+1}\frac{d^2F(\mu)}{d\mu^2} + 2m(1-\mu^2)^{-m/2}\mu\frac{dF(\mu)}{d\mu} + m(m+2)(1-\mu^2)^{-m/2-1}\mu^2F(\mu) \\
- \quad & 2(m+1)\mu\left[(1-\mu^2)^{-m/2}\frac{dF(\mu)}{d\mu} + m\mu(1-\mu^2)^{-m/2-1}F(\mu)\right] \\
+ \quad & [l(l+1)-m(m+1)](1-\mu^2)^{-m/2}F(\mu) = 0 \\
\Longrightarrow \quad & (1-\mu^2)^{-m/2+1}\frac{d^2F(\mu)}{d\mu^2} - 2\mu(1-\mu^2)^{-m/2}\frac{dF(\mu)}{d\mu} + l(l+1)(1-\mu^2)^{-m/2}F(\mu) \\
+ \quad & \left[m(m+2)\frac{\mu^2}{1-\mu^2} - 2(m+1)m\frac{\mu^2}{1-\mu^2}\right](1-\mu^2)^{-m/2}F(\mu) \\
\Longrightarrow \quad & (1-\mu^2)\frac{d^2F(\mu)}{d\mu^2} - 2\mu\frac{dF(\mu)}{d\mu} + [l(l+1) - \frac{m^2}{1-\mu^2}]F(\mu)
\end{aligned}
$$

Therefore we know the solution for $m \neq 0$ case is proportional to $F(\mu)$. We define the following ones called associated Legendre functions:

$$
P_l^m(\mu) = (-1)^m(1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_l(\mu).
$$

**Example 7.4**: Please prove that $\int_{-1}^{+1} P_l^m(\mu)P_l^m(\mu)d\mu$=$\frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}$. Solution:

$$
\begin{aligned}
I_{lm} &= \int_{-1}^{+1} P_l^m(\mu)P_l^m(\mu)d\mu = \int_{-1}^{+1}(1-\mu^2)^m\frac{d^mP_l(\mu)}{d\mu^m}\frac{d^mP_l(\mu)}{d\mu^m}d\mu \\
&= (1-\mu^2)^m\frac{d^mP_l(\mu)}{d\mu^m}\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}} - \int_{-1}^{+1}\frac{d}{d\mu}\left[(1-\mu^2)^m\frac{d^mP_l(\mu)}{d\mu^m}\right]\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}}d\mu \\
&= -\int_{-1}^{+1}\frac{d}{d\mu}\left[(1-\mu^2)^m\frac{d^mP_l(\mu)}{d\mu^m}\right]\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}}d\mu
\end{aligned}
$$

Since we have

$$
\begin{aligned}
&\frac{d}{d\mu}\left[(1-\mu^2)^m \frac{d^m P_l(\mu)}{d\mu^m}\right] = -2m\mu(1-\mu^2)^{m-1}\frac{d^m P_l(\mu)}{d\mu^m} + (1-\mu^2)^m \frac{d^{m+1}P_l(\mu)}{d\mu^{m+1}} \\
&(1-\mu^2)\frac{d^{m+1}P_l(\mu)}{d\mu^{m+1}} = 2m\mu\frac{d^m P_l(\mu)}{d\mu^m} - [l(l+1)-m(m-1)]\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}} \\
&I_{lm} = [l(l+1)-m(m-1)]\int_{-1}^{+1}(1-\mu^2)^{m-1}\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}}\frac{d^{m-1}P_l(\mu)}{d\mu^{m-1}}d\mu \\
= \;& (l+m)(l-m+1)I_{l,m-1}.
\end{aligned}
$$

Naturally we have

$$
\begin{aligned}
I_{lm} &= (l+m)(l-m+1)I_{l,m-1} = (l+m-1)(l-m+2)I_{l,m-2} \\
&= (l+m)(l+m-1)...(l+1)l...(l-m+2)(l-m+1)I_{l,0} = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}.
\end{aligned}
$$

**Example 7.5**: Please prove that (a) $(l-m+1)\sqrt{1-\mu^2}P_l^{m-1}{=}P_{l-1}^m - \mu P_l^m$.

(b)$(2l+1)\sqrt{1-\mu^2}P_l^{m-1}{=}P_{l-1}^m - P_{l+1}^m$

(c)$(2l+1)\mu P_l^m{=}(l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m$

Solution:

(a):

From $lP_l(\mu){=}\mu P_l'(\mu) - P_{l-1}'(\mu)$. Taking $m-1$ times derivatives:

$$
\begin{aligned}
l\frac{d^{m-1}P_l}{d\mu^{m-1}} &= \mu\frac{d^m P_l}{d\mu^m} + (m-1)\frac{d^{m-1}P_l}{d\mu^{m-1}} - \frac{d^m P_{l-1}}{d\mu^m}. \\
(l-m+1)(-1)^m(1-\mu^2)^{m/2}\frac{d^{m-1}P_l}{d\mu^{m-1}} &= \mu(-1)^m(1-\mu^2)^{m/2}\frac{d^m P_l}{d\mu^m} - (1-\mu^2)^{m/2}(-1)^m\frac{d^m P_{l-1}}{d\mu^m}. \\
(l-m+1)\sqrt{1-\mu^2}(-1)P_l^{m-1} &= \mu P_l^m - P_{l-1}^m. \\
\Longrightarrow (l-m+1)\sqrt{1-\mu^2}P_l^{m-1} &= -\mu P_l^m + P_{l-1}^m.
\end{aligned}
$$

(b):

From $(2l+1)P_l{=}P_{l+1}' - P_{l-1}'$, taking $m-1$ times derivatives:

$$
\begin{aligned}
(2l+1)\frac{d^{m-1}P_l}{d\mu^{m-1}} &= \frac{d^m P_{l+1}}{d\mu^m} - \frac{d^m P_{l-1}}{d\mu^m} \\
(2l+1)(1-\mu^2)^{m/2}(-1)^{m-1}\frac{d^{m-1}P_l}{d\mu^{m-1}} &= (1-\mu^2)^{m/2}(-1)^{m-1}\frac{d^m P_{l+1}}{d\mu^m} - (1-\mu^2)^{m/2}(-1)^{m-1}\frac{d^m P_{l-1}}{d\mu^m} \\
\Longrightarrow (2l+1)(1-\mu^2)^{1/2}P_l^{m-1} &= P_{l+1}^m - P_{l-1}^m.
\end{aligned}
$$

(c):

From (a) and (b) we have

$$\begin{aligned}(2l+1)P_{l-1}^m - \mu(2l+1)P_l^m &= (l-m+1)P_{l-1}^m - (l-m+1)P_{l+1}^m.\\ \mu(2l+1)P_l^m &= -(l-m+1)P_{l-1}^m + (l-m+1)P_{l+1}^m + (2l+1)P_{l-1}^m\\ \Longrightarrow (2l+1)P_l^m &= (l+m)P_{l-1}^m + (l-m+1)P_{l+1}^m\end{aligned}$$

**Home Work 7.4**: Please show that

(a)$P_l^{m+1}(\mu)$=$-m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m(\mu) - \sqrt{1-\mu^2}\frac{d}{d\mu}P_l^m(\mu)$.

(b)$(l+m)(l-m+1)P_l^{m-1}(\mu)$=$\sqrt{1-\mu^2}\frac{d}{d\mu}P_l^m(\mu) - m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m$.

## IV. SPHERICAL HARMONICS

The general solution to Laplace equation in spherical coordinate is written as

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l} l\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)P_l^m(\cos\theta)e^{im\phi}.$$

For convenience we define the spherical harmonics $Y_{lm}(\theta,\phi)$:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}.$$

With such definition we have

$$\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} Y_{lm}(\theta,\phi)Y_{l'm'}^*(\theta,\phi) = \delta_{ll'}\delta_{mm'}.$$

**Example 7.6** Find the solution to $\nabla^2 V(r,\theta,\phi)$=0 for $r < a$ with the boundary conditions: $V(r=a,\theta,0\leq\phi\leq\pi)$=$V_0$, $V(r=a,\theta,\pi\leq\phi\leq 2\pi)$=0.

Solution:

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} A_{lm}r^l Y_{lm}(\theta,\phi).$$

$$\begin{aligned}V(a,\theta,\phi) &= V_0,\ \ 0\leq\phi\leq\pi,\\ &= 0,\ \ \pi\leq\phi\leq 2\pi.\end{aligned}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}a^l Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta,\phi)d\Omega = V_0\int_{-1}^{+1} d\cos\theta\int_0^{\pi} d\phi Y^*_{l'm'}(\theta\phi).$$

$$\begin{aligned}
A_{l'm'}a^{l'} &= V_0\sqrt{\frac{(2l'+1)(l'-m')!}{4\pi(l'+m')!}}\int_{-1}^{+1} P_{l'}^{m'}(\cos\theta)d\cos\theta\int_0^{\pi} e^{-m'\phi}d\phi \\
A_{l'm'} &= \frac{V_0}{a^{l'}}\sqrt{\frac{(2l'+1)(l'-m')!}{4\pi(l'+m')!}}\frac{2}{im'}\int_{-1}^{+1} P_{l'}^{m'}(\cos\theta)d\cos\theta,\ \ odd\ m' \\
A_{l'm'} &= 0,\ \ even\ m'.
\end{aligned}$$

Set $C_{lm}$=$\int_{-1}^{+1} P_l^m(\cos\theta)d\cos\theta$. For $l$=$m$=0, one has $A_{00}$=$\frac{V_0}{2}$. For $m=0$, but $l\neq 0$. we has $\int_{-1}^{+1} p_l(\cos\theta)\cos\theta$=0. Therefore we have

$$V(r,\theta,\phi) = \frac{V_0}{2} + \sum_{l=1}^{\infty}\sum_{odd\, m=-l}^{l}\left(\frac{r}{a}\right)^l\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\frac{2}{im}Y_{lm}(\theta\phi).$$

**Home Work 7.5**Please prove that $C_{lm}$=$\pi P_{l+1}(0)P_l^m(0)\frac{m}{l}(-1)^{(m+1)/2}$.

## V. THE ADDITION THEOREM

$\gamma$ is the angle between two vectors $\vec{r}$=$r\sin\theta\cos\phi\hat{e}_x + r\sin\theta\sin\phi\hat{e}_y + r\cos\theta\hat{e}_z$ and $\vec{r'}$=$r\sin\theta'\cos\phi'\hat{e}_x + r\sin\theta'\sin\phi'\hat{e}_y + r\cos\theta'\hat{e}_z$. one has

$$P_l(\cos\gamma) = \sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta,\phi).$$

**Proof**:

$P_l(\cos\gamma)$ can be expanded as

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_m(\theta',\phi')Y_{lm}(\theta,\phi).$$

Here

$$A_m = \int P_l(\cos\gamma)Y^*_{lm}(\theta,\phi)d\omega.$$

If we try to use the coordinate where $\vec{r'}$ is the $z$-axis, then we can regard $Y^8_{lm}(\theta,\phi)$ as a function of $\gamma$ and $\beta$. Here $\vec{r}$=$r\sin\gamma\cos\beta\hat{e'}_x + r\sin\gamma\sin\beta\hat{e'}_y + r\cos\gamma\hat{e'}_z$.

$$Y^*_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} B_{m'}Y_{lm'}(\gamma,\beta).$$

If $\gamma$ approaches zero, which means $\vec{r}$ moves to $\vec{r'}$ then we know only the term of $m'$=0 survives because of symmetry. That is

$$\lim_{\gamma\to 0} Y_{lm}^*(\theta,\phi) = Y_{lm}^*(\theta',\phi') = B_0 Y_{l0}(0,\beta) = B_0\sqrt{\frac{2l+1}{4\pi}}.$$

Hence we have

$$B_0 = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}^*(\theta',\phi').$$

On the other hand we have

$$B_0 = \int Y_{lm}^*(\theta,\phi) Y_{l,0}(\gamma,\beta) d\Omega_\gamma = \sqrt{\frac{2l+1}{4\pi}} \int Y_{lm}^*(\theta,\phi) P_l(\cos\gamma) d\Omega_\gamma = \sqrt{\frac{2l+1}{4\pi}} A_m(\theta',\phi').$$

Therefore we have

$$A_m(\theta',\phi') = \frac{4\pi}{2l+1} Y_{lm}^*(\theta',\phi').$$

So we prove this theorem.