Lecture VIII: Speical function: Bessel functions

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This lecture introduce Bessel functions.

I. BESSEL FUNCTIONS

In chapter 1 we have known that Laplace equation in the cylindrical coordinate generate the following differential equation:

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right)+xR-\frac{m^2}{x}R=0.$$

Using the method of Frobenius one obtains one solution:

$$J_m(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{m+2n}.$$

However if m is integer then we cannot obtain two independent solutions, instead one has

$$J_{-m}(x)=(-1)^m J_m(x).$$

Note that if m is not an integer, then J_m and J_{-m} are two independent solutions. To obtain another independent solution in m=integer case, we play a trick to define the following function:

$$N_m(x)=\lim_{\nu\to m}\frac{J_\nu(x)\cos\nu\pi-J_{-\nu}(x)}{\sin\nu\pi}.$$

Actually one can use the conventional method to obtain the second solution. This is left as an home work. To study the behaviour of those functions at very large x one needs to apply the technique developed in Chapter 2. One finds that

$$J_m(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$
$$N_m(x)\sim\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$

The Hankel functions are defined as follows:

$$H^{(1)}(x)_m(x) = J_m(x) + iN_m(x),$$
$$H^{(2)}(x)_m(x) = J_m(x) - iN_m(x),$$

The asymptotic behaviour of Hankel function are

$$H_m^{(1,2)}(x)) \sim \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right].$$

Generating function for Bessel functions:

It won't be difficult to show that

$$G(t, x) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{m=\infty} t^m J_m(x).$$

The way to prove it is simply to make an expansion of t:

$$\begin{aligned}
G(t,x) &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\left(\frac{x}{2}\right)^r \sum_{s=0}^{\infty} \frac{t^{-s}}{s!}\left(\frac{-x}{2}\right)^s \\
&= \sum_{r\geq s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!s!} + \sum_{s>r=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!s!} \\
&= \sum_{n=r-s=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{t^n}{(n+s)!s!} + \sum_{n=s-r=1}^{\infty} \sum_{s=n}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{-n+2s} \frac{t^{-n}}{(-n+s)!s!} \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{t^n}{(n+s)!s!} + \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{-n+2s} \frac{t^{-n}}{(s-n)!s!} \\
&= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=1}^{\infty} \sum_{u=s-n=0}^{\infty} (-1)^{u+n} \left(\frac{x}{2}\right)^{2u+n} \frac{t^{-n}}{u!s!} \\
&= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=1}^{\infty} t^{-n}(-1)^n J_n(x) \\
&= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=1}^{\infty} t^{-n} J_{-n}(x) = \sum_{n=-\infty}^{\infty} t^n J_n(x).
\end{aligned}$$

Orthogonality of $J_m(x)$:

$k_{mn}=\frac{\alpha_{mn}}{a}$

$$\begin{aligned}
& J_m(k_{ml}\rho)\frac{dJ_m(k_{mn}\rho)}{d\rho}|_{\rho=0}^{a} - J_m(k_{mn}\rho)\frac{dJ_m(k_{ml}\rho)}{d\rho}|_{\rho=0}^{a} \\
= & \left(k_{ml}^2 - k_{mn}^2\right)\int_0^a \rho J_m(k_{ml}\rho)J_m(k_{mn}\rho)d\rho. \\
\Longrightarrow & J_m(\xi\rho)\frac{dJ_m(k_{mn}\rho)}{d\rho}|_{\rho=0}^{a} - J_m(k_{mn}\rho)\frac{dJ_m(\xi\rho)}{d\rho}|_{\rho=0}^{a} = \left(\xi^2 - k_{mn}^2\right)\int_0^a \rho J_m(\xi\rho)J_m(k_{mn}\rho)d\rho. \\
\Longrightarrow & J_m(\xi\rho)\frac{dJ_m(k_{mn}\rho)}{d\rho}|_{\rho=a} = \left(\xi^2 - k_{mn}^2\right)\int_0^a \rho J_m(\xi\rho)J_m(k_{mn}\rho)d\rho. \\
\Longrightarrow & a\frac{J_m(\xi\rho)}{d\xi a}\frac{dJ_m(k_{mn}\rho)}{d\rho}|_{\rho=a} = 2\xi\int_0^a \rho J_m(\xi\rho)J_m(k_{mn}\rho)d\rho. + \left(\xi^2 - k_{mn}^2\right)\int_0^a \rho\frac{dJ_m(\xi\rho)}{d\xi}J_m(k_{mn}\rho)d\rho. \\
\xi & \to k_{mn},\ \ [aJ_m(k_{mn}a)]^2 = 2k_{mn}\int_0^a \rho[J_{mn}(k_{mn}\rho)]^2 d\rho \\
\Longrightarrow & \int_0^a \rho[J_m(k_{mn}rho)]^2 d\rho = \frac{a^2}{2}[J_m'(k_{mn}a)]^2.
\end{aligned}$$

Recursion relations:

Example 8.1: Please show that (a) $\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right)=-\frac{J_{m+1}(x)}{x^m}$. (b) $\frac{d}{dx}[x^m J_m(x)]=x^m J_{m-1}(x)$. (c) $J_{m+1}(x)+J_{m-1}(x)=\frac{2m}{x}J_m(x)$. (d)$J_{m+1}(x)-J_{m-1}(x)=-2\frac{dJ_m(x)}{dx}$

(a):

Solution:

From

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} t^m J_m(x),$$

we set t as $\frac{s}{x}$ then

$$\exp\left[\frac{x}{2}\left(\frac{s}{x}-\frac{x}{s}\right)\right] = \sum_{m=-\infty}^{\infty} s^m\frac{J_m(x)}{x^m}.$$

. Now taking derivative on x:

$$\begin{aligned}
\frac{d}{dx}\left(\exp\left[\frac{x}{2}\left(\frac{s}{x}-\frac{x}{s}\right)\right]\right) &= \sum_{m=-\infty}^{\infty} s^m\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right). \\
\left(-\frac{x}{s}\right)\exp\left[\frac{x}{2}\left(\frac{s}{x}-\frac{x}{s}\right)\right] &= \sum_{m=-\infty}^{\infty} s^m\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right). \\
-\sum_{m=-\infty}^{\infty}\frac{s^{m-1}}{x^{m-1}}J_m(x) &= \sum_{m=-\infty}^{\infty} s^m\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right).
\end{aligned}$$

$$\sum_{m=-\infty}^{\infty} s^m \left[-\frac{J_{m+1}(x)}{x^m} - \frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right)\right] = 0.$$
$$\implies \frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right) = -\frac{J_{m+1}(x)}{x^m}.$$

(b):

Solution:

We set t as sx then

$$\exp\left[\frac{x}{2}\left(sx - \frac{1}{sx}\right)\right] = \sum_{m=-\infty}^{\infty} s^m x^m J_m(x).$$

. Now taking derivative on x:

$$\frac{d}{dx}\left(\exp\left[\frac{x}{2}\left(sx - \frac{1}{sx}\right)\right]\right) = \sum_{m=-\infty}^{\infty} s^m \frac{d}{dx}\left(x^m J_m(x)\right).$$
$$sx \exp\left[\frac{x}{2}\left(sx - \frac{1}{sx}\right)\right] = \sum_{m=-\infty}^{\infty} s^m \frac{d}{dx}\left(x^m J_m(x)\right).$$
$$-\sum_{m=-\infty}^{\infty} s^{m+1} x^{m+1} J_m(x) = \sum_{m=-\infty}^{\infty} s^m \frac{d}{dx}\left(x^m J_m(x)\right).$$
$$\sum_{m=-\infty}^{\infty} s^m \left[x^m J_{m-1}(x) - \frac{d}{dx}\left(x^m J_m(x)\right)\right] = 0.$$
$$\implies \frac{d}{dx}\left(x^m J_m\right) = x^m J_{m-1}(x).$$

(c):

Solution:

$$
\begin{aligned}
\frac{\partial G(t,x)}{\partial t} &= \left(\frac{x}{2}+\frac{x}{2t^2}\right)G = \sum_{m=-\infty}^{\infty} m t^{m-1} J_m(x). \\
\left(1+\frac{1}{2t^2}\right)G(t,x) &= \sum_{m=-\infty}^{\infty} 2mt^{m-1}\frac{J_m(x)}{x}. \\
\sum_{m=-\infty}^{\infty}\left(t^m J_m(x)+t^{m-2}J_m(x)\right) &= \sum_{m=-\infty}^{\infty} 2mt^{m-1}\frac{J_m(x)}{x}. \\
\Longrightarrow \sum_{m=-\infty}^{\infty} t^{m-1}J_{m-1}(x)+t^{m-1}J_{m+1}(x) &= \sum_{m=-\infty}^{\infty} t^{m-1}\frac{2mJ_m(x)}{x}. \\
\sum_{m=-\infty}^{\infty} t^{m-1}\left[J_{m-1}(x)+J_{m+1}(x)-\frac{2m}{x}J_m(x)\right] &= 0. \\
\Longrightarrow J_{m-1}(x)+J_{m+1}(x) &= \frac{2m}{x}J_m(x).
\end{aligned}
$$

(d):

Solution:

$$
\begin{aligned}
\frac{\partial G(t,x)}{\partial x} &= \left(\frac{t}{2}-\frac{1}{2t}\right)G = \sum_{m=-\infty}^{\infty} t^m\frac{dJ_m(x)}{dx}. \\
\left(t-\frac{1}{t}\right)G(t,x) &= \sum_{m=-\infty}^{\infty} 2t^m\frac{dJ_m(x)}{dx}. \\
\sum_{m=-\infty}^{\infty}\left(t^{m+1}J_m(x)-t^{m-1}J_m(x)\right) &= \sum_{m=-\infty}^{\infty} 2t^m\frac{dJ_m(x)}{dx}. \\
\Longrightarrow \sum_{m=-\infty}^{\infty} t^m J_{m-1}(x)-t^m J_{m+1}(x) &= \sum_{m=-\infty}^{\infty} t^m 2\frac{dJ_m(x)}{dx}. \\
\sum_{m=-\infty}^{\infty} t^m\left[J_{m-1}(x)-J_{m+1}(x)-2x\frac{dJ_m(x)}{dx}\right] &= 0. \\
\Longrightarrow J_{m-1}(x)-J_{m+1}(x) &= 2\frac{dJ_m(x)}{dx}.
\end{aligned}
$$

Example 8.2:Please show that $J_m(kr)=\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{ikr\sin\phi-im\phi}$.

Solution:

From $\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{m=-\infty}^{\infty} t^m J_m(x)$ we set $x=kr$ and $t=e^{i\phi}$ then

$$\begin{aligned}
\exp\left[\frac{kr}{2}\left(e^{i\phi}-e^{-i\phi}\right)\right] &= \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(kr). \\
\exp\left[ikr\sin\phi\right] &= \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(kr). \\
\int_0^{2\pi} \exp\left[ikr\sin\phi\right] e^{-in\phi} d\phi &= \sum_{m=-\infty}^{\infty} J_m(kr) \int_0^{2\pi} d\phi e^{i(m-n)\phi}. \\
\int_0^{2\pi} \exp\left[ikr\sin\phi\right] e^{-in\phi} d\phi &= \sum_{m=-\infty}^{\infty} 2\pi\delta_{nm} J_m(kr). \\
\frac{1}{2\pi}\int_0^{2\pi} \exp\left[ikr\sin\phi\right] e^{-in\phi} d\phi &= J_n(kr)
\end{aligned}$$

Home Work 8.1 Please prove that (a) $\sin x=2\sum_{n=0}^{\infty} J_{2n+1}(x)$. (b) $J_n(x+y)=\sum_{m=-\infty}^{\infty} J_m(x)J_{n-m}(y)$.

Home Work 8.2: (a) Please prove that $\int_0^{\infty} e^{-ax}J_0(x)dx=\frac{1}{\sqrt{a^2+1}}$. (b) Please evaluate $\int_0^{\infty} e^{-ax}J_m(x)dx$.

Home Work 8.3: Please show that $J_n(x)=x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$.

Example 8.3: $\nabla^2 V(\rho,\theta,z)=0$ with the boundary conditions: $V=V_0$ for $z=h$ and $\rho\le a$; $V=0$ for $z=0$ and $\rho\le a$ and for ρ=a and $0\le z\le h$. Please find out the solution for $\rho\le a$ and $0\le z\le h$.

Solution:

The general solution is

$$V(\rho,\theta,z)=\int dk\left(\sum_{m=-\infty}^{\infty} A_m J_m(k\rho)e^{im\theta}\sinh kz + B_m J_m(k\rho)e^{im\phi}\cosh kz\right).$$

Since $V(z=0)$=0, hence B_m=0. Furthermore, V is independent of θ so that m=0. Therefore one has

$$V(\rho,z)=\int dk A_0 J_0(k\rho)\sinh kz$$

Since $V(\rho=a)$=0 such that k cannot be arbitrary numbers. $J_0(ka)$ must be zero. We then know $k=\frac{\alpha_{0n}}{a}$. α_{0n} is the $n-$th zero of $J_0(x)$. Therefore we have

$$V(\rho,z)=\sum_{n=1}^{\infty}A_nJ_0(k_{0n}\rho)\sinh k_{0n}z.$$

The last boundary condition tell us that

$$V(\rho,z=h)=\sum_{n=1}^{\infty}A_nJ_0(k_{0n}\rho)\sinh k_{0n}h=V_0$$

Applying the orthogonality of Bessel functions we integrate the both sides with multiplying $\rho J_0(k_{0l}\rho)$:

$$\begin{aligned}V_0\int_0^a \rho J_0(k_{0l}\rho)d\rho &= \int_0^a d\rho\sum_{n=1}^{\infty}A_n\rho J_0(k_{0l}\rho)J_0(k_{0n}\rho)\sinh k_{0n}h,\\ &= \sum_{n=1}^{\infty}A_n\sinh k_{0n}h\int_0^a d\rho\rho J_0(k_{0l}\rho)J_0(k_{0n}\rho)\\ &= A_r\sinh k_{0r}h\frac{a^2}{2}[\frac{dJ_0(k_{0r}\rho)}{dk_{0r}\rho}|_{\rho=a}]^2\end{aligned}$$

Now we have

$$\int_0^a \rho J_0(k_{0l}\rho)d\rho=\frac{1}{k_{0r}}\int_0^a d\rho\frac{dk_{0r}\rho J_1(k_{0r}\rho)}{dk_{0r}\rho}=\frac{a}{k_{0r}}J_1(k_{0r}a).$$

Hence we have

$$A_r=\frac{2V_0J_1(k_{0r}a)}{a^2k_{0r}[J_1'(k_{0r}a)]^2\sinh k_{0r}h}.$$

Home Work 8.4: Please prove that $J_0(kR)=e^{im\phi}J_m(k\rho)J_m(k\rho')$ here $R=\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}$.

II. MODIFIED BESSEL FUNCTIONS

When we solve Laplace equation in the cylinder coordinate, if we have periodic function along the z direction then the equation for radial coordinate becomes

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right)-xR-\frac{m^2}{x}R=0.$$

The solution is called modified Bessel function which is related to the Bessel function as follows:

$$I_m(x) = \frac{1}{i^m} J_m(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+m)!} \left(\frac{x}{2}\right)^{m+2n} ..$$

Similarly in the negative n case we have,

$$I_{-m}(x) = i^m J_{-m}(ix) = I_m(x).$$

Another independent solution is

$$K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(x).$$

At large x

$$I_m(x) \sim \frac{1}{\sqrt{2\pi x}} e^x, \ K_m(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}.$$

Example 8.4: Find the solution of $\nabla^2 V(\rho, \phi, z)$=0 for $\rho \leq a$ and $0 \leq z \leq h$ with the boundary conditions: V=0 for $\rho \leq a$ and z=0,h. $V(\rho = a, 0 \leq \phi \leq \pi, 0 \leq z \leq h)$=$V_0$ and $V(\rho = a, \pi \leq \phi \leq 2\pi, 0 \leq z \leq h)$=$-V_0$.

Solution:

First we know

$$V(\rho, \phi, z) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi z}{h}\right) I_m\left(\frac{n\pi}{h}\rho\right) e^{im\phi}.$$

$$\begin{aligned} V_0 &= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi z}{h}\right) I_m\left(\frac{n\pi}{h}a\right) e^{im\phi}. \ for, \ 0 \leq \phi \leq \pi, \\ V_0 &= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi z}{h}\right) I_m\left(\frac{n\pi}{h}a\right) e^{im\phi}. \ for, \ \pi \leq \phi \leq 2\pi, \end{aligned}$$

Using the orthogonality of sine functions we have

$$\begin{aligned} A_{nm} \frac{h}{2} I_m\left(\frac{n\pi}{h}a\right) 2\pi &= V_0 \int_0^h dz \left(\int_0^{\pi} d\phi - \int_{\pi}^{2\pi} d\phi\right) \sin\left(\frac{n\pi z}{h}\right) e^{im\phi}, \\ &= \frac{V_0 h}{n\pi}(1 - \cos n\pi) \frac{2[1-(-1)^m]}{im} \\ &= V_0 \frac{-i8h}{n\pi m}, \ for \ odd \, n \ and \ odd \ m, \\ &= 0 \ otherwise. \end{aligned}$$

Example 8.5: Please prove that $\int_0^\infty \rho J_m(k\rho)J_m(k'\rho)d\rho{=}\frac{\delta(k-k')}{k}$.

Solution:

$$J_m(k'\rho)\frac{d}{d\rho}\left(\rho\frac{dJ_m(k\rho)}{d\rho}\right)-J_m(k\rho)\frac{d}{d\rho}\left(\rho\frac{dJ_m(k'\rho)}{d\rho}\right)=[k^2-k'^2]\rho J_m(k'\rho)J_m(k\rho).$$

$$\rho\left[J_m(k'\rho)\frac{dJ_m(k\rho)}{d\rho}-J_m(k\rho)\frac{dJ_m(k'\rho)}{d\rho}\right]_{\rho=0}^{\infty}=[k'^2-k^2]\int_0^\infty d\rho\rho J_m(k'\rho)J_m(k\rho).$$

$$\Longrightarrow \int_0^\infty d\rho\rho J_m(k'\rho)J_m(k\rho)=\frac{1}{k'^2-k^2}\lim_{\rho\to\infty}\rho\left[J_m(k'\rho)\frac{dJ_m(k\rho)}{d\rho}-J_m(k\rho)\frac{dJ_m(k'\rho)}{d\rho}\right].$$

From

$$\frac{dJ_m(k\rho)}{d\rho}=\frac{k}{2}(J_{m-1}(k\rho)-J_{m+1}(k\rho)).$$

and from the large x behaviour of Bessel functions, we have

$$\begin{aligned}
&\lim_{\rho\to\infty}\rho J_m(k'\rho)\frac{dJ_m(k\rho)}{d\rho}=\lim_{\rho\to\infty}\frac{\rho}{2k}[J_m(k'\rho)J_{m-1}(k\rho)-J_m(k'\rho)J_{m+1}(k\rho)]\\
=&\ \frac{k}{2}\frac{2}{\pi}\frac{1}{\sqrt{k'k}}\cos\left(k'\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)\cos\left(k\rho-\frac{(m-1)\pi}{2}-\frac{\pi}{4}\right)\\
-&\ \frac{k}{2}\frac{2}{\pi}\frac{1}{\sqrt{k'k}}\cos\left(k'\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)\cos\left(k\rho-\frac{(m+1)\pi}{2}-\frac{\pi}{4}\right)\\
=&\ \frac{k}{2\pi\sqrt{kk'}}[\cos[(k+k')\rho-m\pi]+\cos[(k-k')\rho+\pi/2]\\
-&\ \cos[(k+k')\rho-(m+1)\pi]-\cos[(k-k')\rho-\pi/2]]\\
=&\ \frac{k}{2\pi\sqrt{kk'}}[\cos[(k+k')\rho-m\pi]-2\sin[(k-k')\rho]-\cos[(k+k')\rho-(m+1)\pi]]
\end{aligned}$$

$$\begin{aligned}
&\lim_{\rho\to\infty}\rho J_m(k\rho)\frac{dJ_m(k'\rho)}{d\rho}\\
=&\ \frac{k'}{2\pi\sqrt{kk'}}[\cos[(k+k')\rho-m\pi]-2\sin[(k'-k)\rho]-\cos[(k+k')\rho-(m+1)\pi]]
\end{aligned}$$

Hence

$$
\begin{aligned}
&\lim_{\rho\to\infty}\rho\left[J_m(k'\rho)\frac{dJ_m(k\rho)}{d\rho}-J_m(k\rho)\frac{dJ_m(k'\rho)}{d\rho}\right]\\
=&\ \frac{k}{2\pi\sqrt{kk'}}[\cos[(k+k')\rho-m\pi]-2\sin[(k-k')\rho]-\cos[(k+k')\rho-(m+1)\pi]]\\
-&\ \frac{k'}{2\pi\sqrt{kk'}}[\cos[(k+k')\rho-m\pi]-2\sin[(k'-k)\rho]-\cos[(k+k')\rho-(m+1)\pi]]\\
=&\ \frac{1}{\pi\sqrt{kk'}}[(k'-k)(-1)^m\cos[(k+k')\rho]-(k'+k)\sin[(k-k')\rho]]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{k'^2-k^2}\lim_{\rho\to\infty}\rho\left[J_m(k'\rho)\frac{dJ_m(k\rho)}{d\rho}-J_m(k\rho)\frac{dJ_m(k'\rho)}{d\rho}\right]\\
=&\ \frac{1}{\pi\sqrt{kk'}}\left[(-1)^m\frac{\cos[(k+k')\rho]}{k+k'}-\frac{\sin[(k-k')\rho]}{k-k'}\right]
\end{aligned}
$$

Because

$$
\begin{aligned}
&\lim R\to\infty\int_{-R}^{R}e^{ikr}dr=2\pi\delta(k)\Longrightarrow\lim R\to\infty\frac{e^{ikR}-e^{-ikR}}{2ik\pi}\\
=&\ \frac{\sin kR}{k\pi}=\delta(k)\Longrightarrow\frac{1}{\sqrt{kk'}}\delta(k-k')=\frac{\delta(k-k')}{k}.
\end{aligned}
$$

and

$$
\lim R\to\infty\int_{-R}^{R}[e^{ikr}+e^{-ikr}]dr\to 0.
$$

Therefore we prove that $\int_0^\infty\rho J_m(k\rho)J_m(k'\rho)d\rho=\frac{\delta(k-k')}{k}$.

Example 8.6:Find the solution of $\nabla^2V(\rho,\phi,z)=0$ for $z>0$ with the boundary condition: $V(\rho,\phi,z=0)=V_0\left(\frac{a}{\rho}\right)\sin(\rho/a)$.

Solution:

$$
V(\rho,\phi,z)=\sum_{m=-\infty}^{\infty}\int_0^\infty dkA_m(k)e^{-kz}J_m(k\rho).
$$

The boundary condition tell us that

$$
V(\rho,\phi,z=0)=\sum_{m=-\infty}^{\infty}\int_0^\infty dkA_m(k)J_m(k\rho)=V_0\left(\frac{a}{\rho}\right)\sin(\rho/a).
$$

It is clear V is independent of ϕ so we know that

$$\int_0^\infty dk A_0(k) J_0(k\rho) = V_0 \left(\frac{a}{\rho}\right) \sin(\rho/a).$$

Using the result of Example 8.5 we have

$$A_0(k) = kV_0 \int_0^\infty \left(\frac{a}{\rho}\right) \sin(\rho/a) J_0(k\rho)\rho d\rho.$$

Home Work 8.5: Please show that $A_0(k)$=$\frac{a^2V_0k}{\sqrt{1-(ka)^2}}$ if $ka < 1$, otherwise $A_0(k)$=0.

III. SPHERICAL BESSEL FUNCTIONS

When one tries to solve wave equation in spherical coordinate, one often meets the following equation:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 - l(l+1) = 0.$$

This equation can be transformed to Bessel equation. If we set $R(r)$=$F(r)r^{-1/2}$. Then,

$$\begin{aligned}
&\frac{dR}{dr} = -\frac{1}{2}F(r)r^{-3/2} + \frac{dF}{dr}r^{-1/2}.\\
&\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{4}Fr^{-1/2} + r^{1/2}\frac{dF}{dr} + r^{3/2}\frac{d^2F}{dr^2}.\\
\Longrightarrow\ &-\frac{1}{4}Fr^{-1/2} + r^{1/2}\frac{dF}{dr} + r^{3/2}\frac{d^2F}{dr^2} + k^2r^{3/2}F - l(l+1)Fr^{-1/2} = 0.\\
\Longrightarrow\ &-\frac{1}{4}F + r\frac{dF}{dr} + r^2\frac{d^2F}{dr^2} + k^2r^2F - l(l+1)F = 0.\\
\Longrightarrow\ &\frac{d}{dr}(r\frac{dF}{dr}) + k^2rF - \left(l+\frac{1}{2}\right)^2\frac{F}{r} = 0.\\
\Longrightarrow\ &R(r) = r^{-1/2}J_{l+1/2}(kr)
\end{aligned}$$

We define the spherical bessel function as follows,

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x) = \frac{\sqrt{\pi}}{2}\sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma\left(n+l+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{l+2n}.$$

Similarly we define spherical Neumann function and Hankel functions:

$$n_l(x) = \sqrt{\frac{\pi}{2x}}N_{l+1/2}(x),\ h_l^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}}H_l^{(1,2)}(x).$$

Home Work 8.6: Please show that the solution of wave equation in spherical coordinate can be written as $F(r,\theta,\phi,t)$=$\sum_{l=0}^\infty\sum_{m=-l}^l h_l^{(1)}(kr)P_l^m(\cos\theta)e^{im\phi}e^{-ikvt}$.

Home Work 8.7 The Frensel integral $S(x){=}\sqrt{\frac{2}{\pi}}\int_0^x dt\sin t^2$. Please show that $S(x){=}\sqrt{\frac{2}{\pi}}x\sum_{n=1}^{\infty} j_{2n-1}(x^2)$.