Mechanics: Lagrangian Mechanics

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

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I. WHY YOU NEED SOMETHING MORE THAN NEWTONIAN MECHANICS?

Why you need take this course "Basic Mechanics" after many years of eduction of " Mechanics"? The simple answer is simply because it is not enough. What you have learned is called Newtonian Mechanics. It is a great system invited by Isaac Newton. However the core of whole system is the concept of "force" becomes, unfortunately, obstacle to apply to other category of physics.

The most pronounced example is theory of relativity. Not only because the concept of "force" violates the principle of special relativity which asserts no speed can be faster than speed of light, but also this concept closely relates with the essence of space which must be assumed to be flat. Hence one definitely cannot apply Newtonian mechanics to Relativity. Another example is Quantum Mechanics. It is easy to see that without introducing Hamiltonian it is almost impossible to learn Quantum Mechanics. Similar situations occur when one try to learn electrodynamics and Statistics Mechanics.

Another more practical reason is as follows: sometimes one simply cannot solve the problem of the system under constraints. To solve the equation of motion one needs know the explicit forms of the constrain forces whose presence guarantee the constraints. Ironically they will be known only is the solutions of the equations of motion is available! Therefore it is necessary to develop another framework which is able to re-generate all result of Newtonian Mechanics and to apply to the situations where Newtonian Mechanics fails.

Fortunately such a framework has been invited and as as a matter of fact, it is as old as Newtonian Mechanics. The great rival of Isaac Newton, Gottfried Wilhelm Leibniz (1646-1716). He argued that the essence of space and time cannot be absolute as Newton insisted, but rather be relative. He started his "Mechanics" from a variation principle. Namely he argued what makes the "actual path" so special? It must be some some reason to distinguish the "actual path" from the "possible paths". Most natural answer is that the "actual path" makes some quantity depending on paths become smallest or largest. He called it "vis viva". Nowadays we call it "action". The mathematical development has been made by Maupertuis, Euler and Lagrangian and Hamilton, it becomes a formidable device. It is definitely worthy of spending one semester to study it. That is the thing exact we will do in this semester.

II. BRACHISTOCHRONE CURVE

The mathematical tool used in Lagrangian Mechanics is called "Calculus of Variation". To understand this method, the best way is look at some concrete example. The most famous example is the problem of "Brachistochrone".

The vertical axis is x-axis and the horizontal axis is y-axis. The initial position is A=$(x = x_1, y = 0)$ and the final position is B=$(x = 0, y = y_0)$, Then the time for a mass point to drop from A to B is

$$T[h(y)] = \int_0^{y_0} dy \sqrt{\frac{1 + \left(\frac{dh(y)}{dy}\right)^2}{2g(x_1 - h(y))}}. \tag{1}$$

here $x = h(y)$ is the path which satisfies $h(0)$=x_1 and $h(y_0)$=0. This formula is derived by the following way. Since the normal force acting on the mass point does no work since this force is always normal to displacement. Therefore only the force is the force of gravity. Therefore we know the energy of mechanical energy is conserved. Namely

$$\frac{m}{2}v^2 + mgx = mgx_1 = constant.$$

Hence $v(x)$=$\sqrt{2g(x_1 - x)}$. In an infinitesimal duration dt the mass point walk the distance ds and we know dt=$\frac{ds}{v}$. Furthermore the distance ds=$\sqrt{(dx)^2 + (dy)^2}$. Therefore

$$T = \int dt = \int \frac{ds}{v(x)} = \int \frac{\sqrt{(dx)^2 + (dy)^2}}{v(x)} = \int_0^{y_0} dy \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(x_1 - x)}}.$$

Set $x = h(y)$ then we obtain Eq.(1). So which path will give the smallest value of T? Euler found a very clever way to solve this kind of problem. Assume the path giving the smallest T is $h_0(y)$. For an arbitrary function $f(x)$ satisfying $f(0)$=$f(y_0)$=0, the function

$h(y)=h_0(y)+\varepsilon f(y)$ also satisfies $h(0)=x_1$ and $h(y_0)=0$ If we choose one particular $f(y)$, T becomes a function of ε. Since the extreme value of $T[\varepsilon]$ occurs at $\varepsilon=0$, i.e. $h(y)=h_0(y)$. Therefore we know $\frac{dT[\varepsilon]}{d\varepsilon}|_{\varepsilon=0}=0$. Now we have

$$T[\varepsilon] = \int_0^{y_0} dy \mathcal{T}\left(h(y), \frac{dh(y)}{dy}\right) = \int_0^{y_0} dy \left(\frac{1+\left(\frac{dh_0(y)}{dy}+\varepsilon\frac{df(y)}{dy}\right)^2}{2g[x_1-(h_0(y)+\varepsilon f(y))]}\right)^{1/2}.$$

The crucial observation is that ε always appears with h and $\frac{dh}{dy}$. Therefore to take derivative with respect to ε one just needs take partial derivative with h and dh/dy and apply the chain rule. Moreover what we need is the derivative with respect ε taking at $\varepsilon=0$. So that we reach

$$\frac{dT[\varepsilon]}{d\varepsilon}|_{\varepsilon=0} = \int_0^{y_0} dy \left[\frac{\partial\mathcal{T}}{\partial h(y)}|_{\varepsilon=0}\frac{\partial h(y)}{\partial\varepsilon} + \frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0}\frac{\partial\frac{dh}{dy}}{\partial\varepsilon}\right]. \tag{2}$$

The next step is very simple. As we define $h(y)=h_0(y) + \varepsilon f(y)$, naturally we have the following,

$$\frac{\partial h}{\partial\varepsilon} = f(y),\ \frac{\partial\frac{dh}{dy}}{\partial\varepsilon} = \frac{df(y)}{dy}. \tag{3}$$

Insert Eq.(3) to Eq. (2) one has

$$\begin{aligned}
\frac{dT[\varepsilon]}{d\varepsilon}|_{\varepsilon=0} &= \int_0^{y_0} dy \left[\frac{\partial\mathcal{T}}{\partial h(y)}|_{\varepsilon=0} f(y) + \frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0}\frac{df}{dy}\right] \\
&= \int_0^{y_0} dy \left[\frac{\partial\mathcal{T}}{\partial h(y)}|_{\varepsilon=0} f(y) + \frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0} f(y)|_{y=0}^{y=y_0} - \int_0^{y_0} dy\frac{d}{dy}\left(\frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0}\right) f(y)\right] \\
&= \int_0^{y_0} dy \left[\frac{\partial\mathcal{T}}{\partial h(y)}|_{\varepsilon=0} - \frac{d}{dy}\left(\frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0}\right)\right] f(y) = 0.
\end{aligned}$$

Note that the surface terms vanish because $f(0)=f(y_0)=0$. Since $f(y)$ is arbitrary function, there is no doubt that the coefficient of $f(y)$ must be identical zero. That means we reach the following result.

$$\frac{\partial\mathcal{T}}{\partial h(y)}|_{\varepsilon=0} - \frac{d}{dy}\left(\frac{\partial\mathcal{T}}{\partial\left(\frac{dh(y)}{dy}\right)}|_{\varepsilon=0}\right) = 0. \tag{4}$$

Eq.(4) is the differential equation which $h_0(y)$ must satisfy. In other words, Euler transformed this complicated extreme value of path into simple differential equation! Marvelous!

In this example we can carry out the following calculation,

$$\frac{\partial \mathcal{T}}{\partial h(y)}|_{\varepsilon=0} = g\left(\frac{1+\left(\frac{dh_0(y)}{dy}+\varepsilon\frac{df(y)}{dy}\right)^2}{[2g(x_1-(h_0(y)+\varepsilon f(y)))]^3}\right)^{1/2}|_{\varepsilon=0} = g\left(\frac{1+\left(\frac{dh_0(y)}{dy}\right)^2}{[2g(x_1-h_0(y))]^3}\right)^{1/2}.$$

and

$$\frac{\partial \mathcal{T}}{\partial \frac{dh(y)}{dy}}|_{\varepsilon=0} = \left(1+\left(\frac{dh_0(y)}{dy}+\varepsilon\frac{df(y)}{dy}\right)^2\right)^{-1/2}(2g(x_1-(h_0(y)+\varepsilon f(y)))^{-1/2}\left(\frac{dh_0}{dy}+\varepsilon\frac{df}{dy}\right)|_{\varepsilon}$$

$$= \left(1+\left(\frac{dh_0(y)}{dy}\right)^2\right)^{-1/2}(2g(x_1-h_0(y))^{-1/2}\left(\frac{dh_0}{dy}\right)$$

At the end we reach the following equation.

$$g\left(\frac{1+\left(\frac{dh_0(y)}{dy}\right)^2}{[2g(x_1-h_0(y))]^3}\right)^{1/2} = \frac{d}{dy}\left[\left(1+\left(\frac{dh_0(y)}{dy}\right)^2\right)^{-1/2}(2g(x_1-h_0(y)))^{-1/2}\left(\frac{dh_0}{dy}\right)\right].$$

From this example we have learned a very powerful way to handle problems of this kind. If the problem is involved with more functions the method is similar. For example, if

$$T[x(y),z(y)] = \int_0^{y_0} dy \mathcal{T}\left(x(y),x(y),\frac{dx}{dy},\frac{dz}{dy},y\right).$$

Then we can set $x(y)$=$q_0(y)$+$\varepsilon_1 f(y)$ and $z(y)$=$p_0(y)$+$\varepsilon_2 g(y)$ here $q_0(y)$ and $p_0(y)$ are the functions generating the extreme value of T and $f(y=0)$=$f(y=y_0)$=$g(y=0)$=$g(y=y_0)$=0. Then we know

$$\frac{\partial T}{\partial \varepsilon_1}|_{\varepsilon_1=0,\varepsilon_2=0} = \frac{\partial T}{\partial \varepsilon_2}|_{\varepsilon_1=0,\varepsilon_2=0} = 0.$$

One can easily obtain the equations $q_0(y)$ and $p_0(y)$ should satisfy:

$$\frac{\partial \mathcal{T}}{\partial x(y)}|_{\varepsilon_1=\varepsilon_2=0} - \frac{d}{dy}\left(\frac{\partial \mathcal{T}}{\partial\left(\frac{dx(y)}{dy}\right)}|_{\varepsilon_1=\varepsilon_2=0}\right) = 0,$$

$$\frac{\partial \mathcal{T}}{\partial z(y)}|_{\varepsilon_1=\varepsilon_2=0} - \frac{d}{dy}\left(\frac{\partial \mathcal{T}}{\partial\left(\frac{dz(y)}{dy}\right)}|_{\varepsilon_1=\varepsilon_2=0}\right) = 0.$$

One can generalize Euler's method to as many degrees of freedom as one likes as long as those degrees of freedom are independent.

III. HAMILTON'S PRINCIPLE

The starting point of Lagrangian Mechanics is the Hamilton's principle. It says that the actual path is the one generating extreme value of the action S. S is defined as

$$S[x(t)] = \int_{t=t_0}^{t=t_1} dtL(x(t), \frac{dx(t)}{dt}, t)dt. \;\; x(t=t_0) = x_0, \;\; x(t=t_1) = x_1.$$

L is a function of $x(t)$, $\frac{dx}{dt}$ and t in general. L is called Lagrangian function. From the previous section, we know Hamilton's principle will derive the following equation for the actual path $x_0(t)$:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right).$$

here $\dot{x}(t)=\frac{dx(t)}{dt}$. This equation should give us the same solution as the one from Newtonian Mechanics. For free body, one is able to deduce $L=\frac{1}{2}m\dot{x}^2$. The equation of motion is

$$\frac{d}{dt}\left(m\dot{x}(t)\right) = 0.$$

Namely one obtains v=constant. Of course it is same with the solution of Newtonian Mechanics. It is straightforward to generalize to two and three dimensional systems. The next thing is to pursuit the Lagrangian of the body under the influence of the external force. Remember that in the case of conservative force the equation of motion is

$$\begin{aligned} m\frac{d^2x(t)}{dt^2} &= -\frac{\partial U}{\partial x}, \\ m\frac{d^2y(t)}{dt^2} &= -\frac{\partial U}{\partial y}, \\ m\frac{d^2z(t)}{dt^2} &= -\frac{\partial U}{\partial z}, \end{aligned}$$

here $U(x, y, z)$ is potential. It is not difficult to find that $L=T-U$ is chosen then one desired equation is derived.

Example 1: $L=\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$. Please derive the equation of motion.

Solution: It is obvious that the equations of motion are

$$m\frac{d^2x(t)}{dt^2} = 0, \;\; m\frac{d^2y(t)}{dt^2} = 0, \;\; m\frac{d^2z(t)}{dt^2} = -mg.$$

Notice that in Lagrangian Mechanics it is not necessary to use Cartesian coordinate. This is very important both for practical and conceptual reasons. Practical reason is best illustrated by the following example.

Example 2: Using Newtonian Mechanics and Lagrangian Mechanics to derive the equation of motion of pendulum.

Solution: From Newtonian Mechanics:

$$m\frac{d^2x(t)}{dt^2} = -F_x,$$
$$m\frac{d^2y(t)}{dt^2} = F_y - mg, F_x/F_y = x/y, x^2+y^2 = l^2.$$

Set x=$l\sin\theta$,y=-$l\cos\theta$. F_x=-$F\sin\theta$, F_y=$F\cos\theta$. So we have

$$m\left(l\frac{d^2\theta}{dt^2}\cos\theta - l\frac{d\theta}{dt}\sin\theta\right) = -F\sin\theta,$$
$$m\left(l\frac{d^2\theta}{dt^2}\sin\theta + l\frac{d\theta}{dt}\cos\theta\right) = F\cos\theta - mg,$$

It is easy to cancel F and obtain

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta.$$

However it is much easier to derive the equation of motion from Lagrangian Mechanics.

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgl(1-\cos\theta) = \frac{m}{2}l^2\dot{\theta}^2 - mgl(1-\cos\theta).$$

From

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta,\ \ \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}.$$

We easily obtain the same equation of motion.

In some case it is even impossible to write solve the problem by Newtonian Mechanics because the force of constraint is unknown. However it cause no trouble in Lagrangian Mechanics. There is one example here.

Example 3: Obtain the equation of motion of a pendulum with the string with variant length $l(t)$=l_0-αt.

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - mg(l_0-\alpha t)(1-\cos\theta).$$

Here x=$(l_0-\alpha t)\sin\theta$ and y=$-(l_0-\alpha t)\cos\theta$ and

$$\dot{x} = (l_0-\alpha t)\dot{\theta}\cos\theta - \alpha\sin\theta,\ \ \dot{y} = (l_0-\alpha t)\dot{\theta}\sin\theta + \alpha\cos\theta.$$

Hence $\dot{x}^2 + \dot{y}^2 = (l_0 - \alpha t)^2\dot{\theta}^2 + \alpha^2$ and sequently

$$L = \frac{m}{2}(l_0 - \alpha t)^2\dot{\theta}^2 + \frac{m}{2}\alpha^2 - mg(l_0 - \alpha t)(1 - \cos\theta).$$

So that we have

$$\frac{\partial L}{\partial \theta} = mg(l_0 - \alpha t)\sin\theta. \ \frac{\partial L}{\partial \dot{\theta}} = m(l_0 - \alpha t)^2\dot{\theta}, \ \ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m(l_0 - \alpha t)^2\ddot{\theta} - 2m(l_0 - \alpha t)\alpha\dot{\theta}.$$

The equation of motion is

$$mg\sin\theta = m(l_0 - \alpha t)\ddot{\theta} - 2m\alpha\dot{\theta}.$$

However I want to emphasize that in the case of nonconservative or velocity-dependent force then one usually find no general way to construct Lagrangian. The most important example is the Lorentz force. One can find some more simple example to see that $L = T - U$ is not applicable.

Example 4: Find L which derives the following equation:

$$m\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$$

Nevertheless one is still able to construct the Lagrangian. The most important guidance is from the symmetry. When the system owns some symmetry. its Lagrangian must also own the same symmetry. In the next lecture we will talk about this point.