Mechanics: Lagrangian Mechanics

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I. EQUIVALENCE OF LAGRANGIANS

Is Lagrangian unique? The answer is NO. There are infinite many Lagrangians giving the same equations of motions. If $L'{=}L + \frac{d}{dt}F(x(t),t)$ with arbitrary function F then it can be seen that L' and L is equivalent because they give the same equation of motion. It can be seen easily by the Hamilton's principle.

$$S' = \int_{t_0}^{t_1} L'(x,\dot{x},t)dt = \int_{t_0}^{t_1} (L(x,\dot{x},t) + \frac{d}{dt}F(x,t))dt$$
$$= \int_{t_0}^{t_1} L(x,\dot{x},t)dt + F(x_1,t_1) - F(x_0,t_0) = S + F(x_1,t_1) - F(x_0,t_0).$$

Since $F(x_1,t_1)$-$F(x_0,t_0)$ is nothing but a constant and independent of the path, therefore the path gives the extreme value of S also gives the extreme value of S'. The associate Euler equation must be same. One can even directly check it out. The Euler equation for S' is

$$\frac{\partial L'}{\partial r_i} = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}_i}\right) \to \frac{\partial L}{\partial r_i} + \frac{\partial}{\partial r_i}\left(\frac{dF(x(t),t)}{dt}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}_i}\frac{dF}{dt}\right)$$
$$\frac{\partial L}{\partial r_i} + \frac{\partial}{\partial r_i}\left(\sum_{k=1}^{3}\frac{\partial F}{\partial r_k}\dot{r}_k + \frac{\partial F}{\partial t}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}_i}\left(\sum_{k=1}^{3}\frac{\partial F}{\partial r_k}\dot{r}_k + \frac{\partial F}{\partial t}\right)\right)$$
$$\frac{\partial L}{\partial r_i} + \left(\frac{\partial^2 F}{\partial r_i \partial r_k}\dot{r}_k + \frac{\partial^2 F}{\partial t \partial r_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial r_i}\right)$$
$$\frac{\partial L}{\partial r_i} + \left(\frac{\partial^2 F}{\partial r_i \partial r_k}\dot{r}_k + \frac{\partial^2 F}{\partial t \partial r_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) + \left(\sum_{k=1}^{3}\frac{\partial^2 F}{\partial r_i \partial r_k}\dot{r}_k + \frac{\partial^2 F}{\partial r_i \partial t}\right)$$
$$\Longrightarrow \frac{\partial L}{\partial r_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right).$$

Indeed they give the same Euler equation.

II. GALILEAN INVARIANCE

In classical physics, we assume the physical law is same for all inertial frames. Namely, if the relative motion between two frames is the constant linear motion, then the observers should observe the same physical law. In more formal language the system should be invariant under the following transform:

$$\vec{r'} = \vec{r} + \vec{v}t, \;\; t' = t.$$

Here $\vec{v}$ is a constant vector. This is called Galilean invariance. We now show the Lagrangian of a free body in one frame is still a free body after Galilean transform. The Lagrangian of a free body is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

Now we replace x, y, x by x', y', z' and obtain

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2\right) - m(v_x\dot{x'} + v_y\dot{y'} + v_z\dot{z'}) + \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) \\ &= \frac{m}{2}\left(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2\right) - m(v_x\dot{x} + v_y\dot{y} + v_z\dot{z}) - \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) \\ &= \frac{m}{2}\left(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2\right) - \frac{d}{dt}\left(mv_x x + v_y y + v_z z + \frac{mt}{2}(v_x^2 + v_y^2 + v_z^2)\right). \end{aligned}$$

Therefore the forms of the Lagrangians in two coordinates are equivalent. It means the object is also free-body in the new coordinate.

The same statement should hold for the body under influence of the external force. In general we consider a two-body system where two bodies interact with each other. Therefore its Lagrangian should be

$$L = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_2^2 + \dot{z}_2^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) - U(x_1, y_1, z_1, x_2, y_2, z_2).$$

To be invariant under Galilean transform U needs to satisfy the certain condition:

$$U(r_i) = U(r_i') + \frac{dF(r_i, t)}{dt}.$$

The most simple way for U to satisfy Galilean invariance is that U only depends on the variables which themselves are invariant under Galilean transforms. It is easy to see the difference of the coordinates of two points are invariant under Galilean transform, so we can assume that

$$U(x_1, y_1, z_1, x_2, y_2, z_2) = U(a = x_2 - x_1, b = y_2 - y_1, c = z_2 - z_1). \tag{1}$$

Consequently we have equations of motion as

$$\begin{aligned} m_1\ddot{x}_1 &= \frac{\partial U}{\partial a}, \;\; m_2\ddot{x}_2 = -\frac{\partial U}{\partial a}, \\ m_1\ddot{y}_1 &= \frac{\partial U}{\partial b}, \;\; m_2\ddot{x}_2 = -\frac{\partial U}{\partial b}, \\ m_1\ddot{z}_1 &= \frac{\partial U}{\partial c}, \;\; m_2\ddot{x}_2 = -\frac{\partial U}{\partial c}. \end{aligned}$$

In term of Newtonian Mechanics, the above equations just tell us the force received by the two body are same in the magnitude but in the opposite directions. Namely it is the third law of Newtonian Mechanics.

III. CENTER OF MASS

One can change variables for the two-body system and reduce this two-body system as two independent one-body systems when U satisfying Eq. (1). Namely we define $\vec{R}$ and $\vec{r}$ as

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \;\; \vec{r} = \vec{r}_1 - \vec{r}_2.$$

Now we can make the following change of the variables:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}, \;\; \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}.$$

It is straightforward to show that

$$\begin{aligned} L &= \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_2^2 + \dot{z}_2^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) - U(x_2 - x_1, y_2 - y_1, z_2 - z_1). \\ &= \frac{m_1 + m_2}{2}\dot{\vec{R}}^2 + \frac{m_1 m_2}{2(m_1 + m_2)}\dot{\vec{r}}^2 - U(r_i). \end{aligned}$$

The corresponding equations of motion are

$$\begin{aligned} M\ddot{R}_i &= 0, \;\; M = m_1 + m_2, \\ \mu\ddot{r}_i &= -\frac{\partial U}{\partial r_i}, \;\; \mu = \frac{m_1 m_2}{m_1 + m_2}. \end{aligned}$$

$\vec{R}$ is the centre of mass of this system and its motion is unform linear motion. On the other hand, the relative coordinate $\vec{r}$ is under the influence of the potential $U(r_i)$. Hence one can neglect the motion of the centre of mass and treat it as a one-body problem under the contain potential.