Mechanics: Lagrangian Mechanics

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

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I. LAGRANGIAN WITH ROTATIONAL INVARIANCE

Here we want to derive the general form of the Lagrangian of a two-body system with rotational invariance. From Galilean invariance we know the Lagrangian of a two-body system should be

$$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Now if the system is invariant under any rotation, then we can choose the following one:

$$x'_1 = x_1 \cos\theta + y_1 \sin\theta,\ \ y'_1 = -x_1 \sin\theta + y_1 \cos\theta,\ \ x'_2 = x_2 \cos\theta + y_2 \sin\theta,\ \ y'_2 = -x_2 \sin\theta + y_2 \cos\theta.$$

It is easy to see that the inverse transform is as follows,

$$x_1 = x'_1 \cos\theta - y'_1 \sin\theta,\ \ y_1 = x'_1 \sin\theta + y'_1 \cos\theta,\ \ x_2 = x'_2 \cos\theta - y'_2 \sin\theta,\ \ y_2 = x'_2 \sin\theta + y'_2 \cos\theta.$$

It is trivial to see that

$$\frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) = \frac{m_1}{2}(\dot{x'}_1^2 + \dot{y'}_1^2 + \dot{z'}_1^2) + \frac{m_2}{2}(\dot{x'}_2^2 + \dot{y'}_2^2 + \dot{z'}_2^2).$$

Therefore we need the potential to satisfy the following condition:

$$U(\Delta x' \cos\theta - \Delta y_1 \sin\theta, \Delta x \sin\theta + \Delta y \cos\theta, \Delta z) = U(\Delta x', \Delta y', \Delta z'). \tag{1}$$

Here $\Delta x' = x'_2 - x'_1$ and $\Delta y' = y'_2 - y'_1$. Now we can make change of variable:

$$U(a, b, c) = U(K \cos\Phi, K \sin\Phi, c) = V(K, \Phi, c).$$

Here K=$\sqrt{a^2+b^2}$ and Φ=$\tan^{-1}\left(\frac{b}{a}\right)$. By this way one can immediately see that Eq.(1) is actually

$$\begin{aligned}
&U(\Delta x', \Delta y', \Delta z') = V(K = \sqrt{\Delta x'^2 + \Delta y'^2}, \Phi = \tan^{-1}\left(\frac{\Delta y'}{\Delta x'}\right), \Delta z),\\
&U(\Delta x' \cos\theta - \Delta y_1' \sin\theta, \Delta x' \sin\theta + \Delta y' \cos\theta, \Delta z')\\
&= U(K\cos\Phi\cos\theta - K\sin\Phi\sin\theta, K\sin\Phi\sin\theta + K\cos\Phi\cos\theta, \Delta z)\\
&= U(K\cos(\Phi+\theta), K\sin(\Phi+\theta), \Delta z)\\
&= V(K = \sqrt{\Delta x'^2 + \Delta y'^2}, \Phi+\theta, \Delta z).
\end{aligned}$$

Since θ is arbitrary, we know V must be independent of Φ! That is

$$U(a,b,c) = V(K = \sqrt{a^2+b^2}, c).$$

We can also choose to make rotation between x and z, y and z. The reason is $U(a,b,c)$=$W(\sqrt{a^2+b^2+c^2})$. In the other words, $U(\Delta x', \Delta y', \Delta z')$=$W(|\vec{r'}_2 - \vec{r'}_1|)$. Now we can find that the equation of motion becomes

$$\begin{aligned}
m\ddot{x}_1 &= -\frac{\partial W}{\partial \xi}\frac{x_1-x_2}{\xi}, m\ddot{y}_1 = -\frac{\partial W}{\partial \xi}\frac{y_1-y_2}{\xi}, m\ddot{z}_1 = -\frac{\partial W}{\partial \xi}\frac{z_1-z_2}{\xi},\\
m\ddot{x}_2 &= -\frac{\partial W}{\partial \xi}\frac{x_2-x_1}{\xi}, m\ddot{y}_2 = -\frac{\partial W}{\partial \xi}\frac{y_2-y_1}{\xi}, m\ddot{z}_2 = -\frac{\partial W}{\partial \xi}\frac{z_2-z_1}{\xi}.
\end{aligned}$$

Here ξ=$\sqrt{|\vec{r'} - \vec{r}}$. It is obvious that not only $\vec{F}_1$=$-\vec{F}_2$ but also $\vec{F}_1 // \vec{r}_2 - \vec{r}_1$. In the other words, the forces between the two bodies along the direction of their relative position. If one calculate the total angular momentum:

$$\begin{aligned}
&\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2\\
&= \left(\vec{R} + \frac{m_2}{m_1+m_2}\vec{r}\right) \times \vec{F}_1 + \left(\vec{R} - \frac{m_1}{m_1+m_2}\vec{r}\right) \times \vec{F}_2\\
&= \vec{R} \times (\vec{F}_1 + \vec{F}_2) + \frac{m_2}{m_1+m_2}\vec{r} \times \vec{F}_1 - \frac{m_1}{m_1+m_2}\vec{r} \times \vec{F}_2\\
&= 0
\end{aligned}$$

From Newtonian Mechanics we know $\vec{\tau}$=$\frac{d\vec{L}}{dt}$. Hence the angular momentum $\vec{L}$ is constant. This is always true for any system with rotational invariance. However, one does not need the concept of force and still know the angular momentum is conserved in the rotational invariant system. This is a very important advantage of Lagrangian Mechanics. The close relation between the conserved observable and the symmetry of the system is manifest in the Lagrangian. This is the topic of the next section.

II. ROTATIONAL INVARIANCE AND CONSERVATION OF ANGULAR MOMENTUM

In this section we will prove that the angular momentum is conserved when the system is invariant. Imagine we make a infinitesimal rotation whose rotation axis is along the direction $\hat{n}$ and the rotation angle is $\delta\phi$. We adopt the notation $\delta\vec{\phi}$=$|\delta\phi|\hat{n}$. Under this rotation a vector $\vec{r}$ becomes $\vec{r'}$ and we can show that $\delta\vec{r}$=$\vec{r'}-\vec{r}$=$\delta\vec{\phi}\times\vec{r}$. Note that we can decompose $\vec{r}$=$\vec{r}_{\parallel}+\vec{r}_{\perp}$. $\vec{r}_{\parallel}$=$(\vec{r}\cdot\hat{n})\hat{n}$. So that $\vec{r}_{\perp}\cdot\hat{n}$=0. Under the rotation $\vec{r}_{\parallel}$ doesn't receive any change. On the contrary, $\vec{r}_{\perp}$ becomes $\vec{r}_{\perp}+|\delta\phi||\vec{r}_{\perp}|\hat{t}$. Here $\hat{t}$ is the tangent vector which is normal to $\vec{r}_{\perp}$ and $\hat{n}$. Therefore we know $\delta\vec{r}$=$|\delta\phi||\vec{r}_{\perp}|\hat{t}$. On the other hand, we have

$$
\begin{aligned}
& \delta\vec{\phi}\times\vec{r} = \delta\vec{\phi}\times(\vec{r}_{\parallel}+\vec{r}_{\perp}) = \delta\vec{\phi}\times\vec{r}_{\perp} \\
= \ & |\delta\phi||\vec{r}_{\perp}|\hat{n}\times\hat{r}_{\perp} = |\delta\phi||\vec{r}_{\perp}|\hat{t} = \delta\vec{r}.
\end{aligned}
$$

Another way to check this result is to use the spherical coordinate. Assume that one makes a rotation along the z axis, which is, $\phi\rightarrow\phi+\delta\phi$.

$$\vec{r} = r\sin\theta\cos\phi\hat{e}_x + r\sin\theta\sin\phi\hat{e}_y + r\cos\theta\hat{e}_z.$$

Assume that one makes a rotation along the z axis, which is, $\phi\rightarrow\phi+\delta\phi$. Since $\delta\phi$ is infinitesimal so $\sin\delta\phi\sim\delta\phi$ and $\cos\delta\phi\sim 1$.

$$
\begin{aligned}
& \vec{r'} = r\sin\theta\cos(\phi+\delta\phi)\hat{e}_x + r\sin\theta\sin(\phi+\delta\phi)\hat{e}_y + r\cos\theta\hat{e}_z. \\
= \ & r\sin\theta(\cos\phi-\sin\phi\delta\phi)\hat{e}_x + r\sin\theta(\sin\phi+\cos\phi\delta\phi)\hat{e}_y + r\cos\theta\hat{e}_z.
\end{aligned}
$$

Such that

$$\begin{aligned} \delta\vec{r} &= r\delta\phi(-\sin\theta\sin\phi\hat{e}_x + \sin\theta\cos\phi\hat{e}_y) \\ &= \delta\phi\hat{e}_z \times (r\sin\theta\cos\phi\hat{e}_x + r\sin\theta\sin\phi\hat{e}_y + r\cos\theta\hat{e}_z) \\ &= \delta\vec{\phi}\times\vec{r}. \end{aligned}$$

If a system is invariant under an infinitesimal rotation then we know

$$\delta L = L(\vec{r}+\delta\vec{r},\dot{\vec{r}}+\delta\dot{\vec{r}},t) - L(\vec{r},\dot{\vec{r}},t) = 0.$$

The next step is

$$\delta L = \sum_{i=1}^{3}\left(\frac{\partial L}{\partial r_i}\delta r_i + \frac{\partial L}{\partial \dot{r}_i}\delta\dot{r}_i\right).$$

Apply the equation of the motion we obtain,

$$\delta L = \sum_{i=1}^{3}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i}\delta r_i + \frac{\partial L}{\partial \dot{r}_i}\delta\dot{r}_i\right).$$

We define the momentum of the system in the following way:

$$p_i = \frac{\partial L}{\partial \dot{r}_i}.$$

So we have

$$\begin{aligned} \delta L &= \left(\frac{d\vec{p}}{dt}\cdot\delta\vec{r}+\vec{p}\cdot\frac{\delta\vec{r}}{dt}\right) = \left(\frac{d\vec{p}}{dt}\cdot\delta\vec{\phi}\times\vec{r}+\vec{p}\cdot\delta\phi\times\frac{d\vec{r}}{dt}\right) \\ &= \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{d\vec{p}}{dt}+\frac{d\vec{r}}{dt}\times\vec{p}\right) = \delta\vec{\phi}\cdot\frac{d}{dt}(\vec{r}\times\vec{p}) = 0. \end{aligned}$$

Since $\delta\vec{\phi}$ is arbitrary, so we know $\vec{r}\times\vec{p}$ must be a constant.

III. TRANSLATION INVARIANCE AND CONSERVATION OF MOMENTUM AND ENERGY

We have learned that some physical observable will be conserved when the system is invariant under certain transform. This is actually a very general thing. Here we will provide two more examples. One is translation and another one is temporal translation. If the system is invariant under this transform: $\vec{r} \to \vec{r} + \vec{a}$ here $\vec{a}$ is an infinitesimal arbitrary constant vector. Then

$$\delta L = L(\vec{r} + \vec{a}, \dot{\vec{r}}, t) - L(\vec{r}, \dot{\vec{r}}, t) = 0.$$

Naturally we have

$$\delta L = \sum_{i=1}^{3} \frac{\partial L}{\partial r_i} a_i = \sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) a_i = \frac{d\vec{p}}{dt} \cdot \vec{a} = 0.$$

Hence $\vec{p}$ is a constant. From this one learns that the momentum is conserved if the system is translation invariant. If the system only has translation invariant in one certain direction, then it is clear that only the momentum in that particular direction is conserved. The next example is temporal translational invariance. Namely it means if the system remains the same when $t \to t + \delta t$. What kind of physical observable will be conserved? We start from the following identity:

$$\begin{aligned}
\frac{dL}{dt} &= \sum_{i=1}^{3}\left(\frac{\partial L}{\partial r_i}\dot{r}_i + \frac{\partial L}{\partial \dot{r}_i}\ddot{r}_i\right) + \frac{\partial L}{\partial t} \\
&= \sum_{i=1}^{3}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right)\dot{r}_i + \frac{\partial L}{\partial \dot{r}_i}\ddot{r}_i\right) + \frac{\partial L}{\partial t} \\
&= \sum_{i=1}^{3}\left(\frac{dp_i}{dt}\dot{r}_i + p_i\ddot{r}_i\right) + \frac{\partial L}{\partial t} \\
&= \frac{d}{dt}(\dot{r}_i p_i) + \frac{\partial L}{\partial t}
\end{aligned}$$

Therefore if $\frac{\partial L}{\partial t}$=0, we have

$$\frac{d}{dt}\left(\sum_{i=1}^{3} p_i\dot{r}_i - L\right) = 0.$$

Hence h=$\sum_{i=1}^{3} p_i\dot{r}_i - L$ will be conserved. Furthermore, if the system is under the influence of conservative force which is independent of velocity, that is, L=$T(r_i) - U(r_i)$. If so

$$\sum_{i=1}^{3} p_i\dot{r}_i = m\sum_{i=1}^{3} v_i^2 = 2T.$$

Then we have h=$2T-(T-U)$=$T+U$. It is the sum of kinetic energy plus potential energy. In Newtonian Mechanics, this is called "Conservation of mechanical energy".

Question: A particle of mass m with speed $\vec{v}$ leave half-space in which its potential energy is a constant U_1 and enters another in which the potential energy is a different constant U_2. Determine change in the direction of motion of the particle.