Mechanics: Lagrangian Mechanics

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

(Dated: October 8, 2012)

This lecture was given on September 27th.

I. PENDULUM IN THE ACCELERATED CAR

There is a pendulum hang at the fixed point inside a car which runs with the constant acceleration a along the horizontal direction. Please find the equation of motion of this pendulum in the inertial frame. The Lagrangian is written as,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mg(l - y).$$

Here x and y is the Cartesian coordinates of the inertial frame. Now we can express them as

$$x = v_0 t + \frac{a}{2}t^2 + l\sin\theta, y = l\cos\theta.$$

Because the motion of pendulum is under the constraint that the string length is constant l. Hence we have,

$$\begin{aligned} L &= \frac{m}{2}((v_0 + at + l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2) - mgl(1 - \cos\theta) \\ &= \frac{ml^2}{2}\dot{\theta}^2 + ml(v_0 + at)\dot{\theta}\cos\theta - mgl(1 - \cos\theta) + \frac{m}{2}(v_0 + at)^2. \end{aligned}$$

The equation of motion is obtained as

$$\begin{aligned} &\frac{d}{dt}\left(ml^2 dot\theta + ml^2(v_0 + at)\cos\theta\right) = -mgl\sin\theta - ml(v_0 + at)\dot{\theta}\sin\theta. \\ \Longrightarrow\ &ml^2\ddot{\theta} - m(v_0 + at)l\dot{\theta}\sin\theta + ma\cos\theta = -mgl\sin\theta - m(v_0 + at)l\dot{\theta}\sin\theta. \\ &ml^2\ddot{\theta} = -mgl\sin\theta - mal\cos\theta. \end{aligned}$$

This is the solution.

II. LAGRANGIAN OF FREE BODY IN THE NON-INERTIAL FRAME

The one advantage of Lagrangian Mechanics is that one does not need the concept of the inertial frame at all. What one concerns is which Lagrangian can provides the correct equation of motion in his own frame. For example, if someone observes a free body with constant speed, then he is able to use the following Lagrangian,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

So what kind of Lagrangian should the observer who sits on the frame with relative acceleration with the original frame use? It is very simple. One can simple carries out the change of variables as follows,

$$x' = x + v_{0x}t + \frac{a_x}{2}t^2,\ \ y' = y + v_{0y}t + \frac{a_y}{2}t^2,\ \ z' = z + v_{0z}t + \frac{a_z}{2}t^2.$$

x', y', z' are the Cartesian coordinates of the new frame. Naturally we have the following relations,

$$\dot{x'} = \dot{x} + v_{0x} + a_x t,\ \dot{y'} = \dot{y} + v_{0y} + a_y t,\ \dot{z'} = \dot{z} + v_{0z} + a_z t,$$

Just insert new variables in the Lagrangian, then one has the Lagrangian in term of $x'y'z'$,

$$\begin{aligned} L &= \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - m(v_{0x}\dot{x'} + v_{0y}\dot{y'} + v_{0z}\dot{z'}) - mt(a_x\dot{x'} + a_y\dot{y'} + a_z\dot{z'}) \\ &- \frac{m}{2}((v_{0x} + a_x t)^2 + (v_{0y} + a_y t)^2 + (v_{0z} + a_z t)^2) \\ &= \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - mt(a_x\dot{x'} + a_y\dot{y'} + a_z\dot{z}) \\ &- \frac{d}{dt}\left[m((v_{0x}x' + v_{0y}y' + v_{0z}z') + \frac{m}{6}\left(\frac{(v_{0x} + a_x t)^3}{a_x} + \frac{(v_{0y} + a_y t)^3}{a_y} + \frac{(v_{0z} + a_z t)^3}{a_z}\right)\right] \end{aligned}$$

Note that Lagrangian is essentially equivalent to another one by adding a total derivative term, so that one has,

$$L = \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - mt(a_x\dot{x'} + a_y\dot{y'} + a_z\dot{z}).$$

The equations of motion become

$$\frac{d}{dt}(m\dot{x'} - ma_x t) = 0,\ \frac{d}{dt}(m\dot{y'} - ma_y t) = 0,\ \frac{d}{dt}(m\dot{z'} - ma_z t) = 0.$$

Therefore in the new frame the "free body" is accelerating. This is of no surprise.

$$m\ddot{x'} = ma_x,\ \ m\ddot{y'} = ma_y,\ \ m\ddot{z'} = ma_z.$$

One can transform Lagrangian in another form,

$$\begin{aligned} L &= \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - mt(a_x\dot{x'} + a_y\dot{y'} + a_z\dot{z}) \\ = &= \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - \frac{d}{dt}\left(mt(a_x x' + a_y y' + a_z z')\right) + m(a_x x' + a_y y' + a_z z') \\ \Longrightarrow L' &= \frac{m}{2}(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) + m(a_x x' + a_y y' + a_z z') \end{aligned}$$

It is easy to see that the corresponding equations of motion are the same.

$$m\ddot{x'} = ma_x, \;\; m\ddot{y'} = ma_y, \;\; m\ddot{z'} = ma_z.$$

III. EXAMPLE: ACCELERATED INCLINED PLANE

One cannot overemphasize the importance of judicious choice of the adequate coordinates. Here is a good example: If a falling body with mass m_2 is falling along the incline plane which has mass m_1 and the angle of incline is θ. The incline plane is on a table without any friction. The falling body will fall along the slide, however the incline plane also moves. Hence it is not a trivial problem in Newtonian Mechanics. Let's try to use Lagrangian Mechanics to solve it. The Lagrangian of the system is given as,

$$\begin{aligned} L &= \frac{m_1}{2}\dot{a}^2 + \frac{m_2}{2}[(\dot{a} + \dot{b}\cos\theta)^2 + (\dot{b}\sin\theta)^2] - m_2 g b\sin\theta \\ &= \frac{m_1}{2}\dot{a}^2 + \frac{m_2}{2}\dot{a}^2 + \frac{m_2}{2}\dot{b}^2 + m_2\dot{a}\dot{b}\cos\theta - m_2 g b\sin\theta. \end{aligned}$$

a is the distance from the origin to the edge of the plane. b is the distance from the edge of the plane to the falling body. Therefore we have following equations of motion,

$$m_1\ddot{a} + m_2\ddot{a} + m_2\ddot{b}\cos\theta = 0,$$

$$m_2\ddot{b} + m_2\ddot{a}\cos\theta = -m_2 g\sin\theta.$$

Multiplying the second equation by the factor $\cos\theta$ then subtract the first equation one obtains,

$$(m_1 + m_2\sin^2\theta)\ddot{a} = m_2 g\sin\theta\cos\theta.$$

Hence we have

$$\ddot{a} = \frac{m_2 g \sin\theta\cos\theta}{m_1 + m_2\sin\theta}, \ \ddot{b} = \frac{(m_1+m_2)g\sin\theta}{m_1 + m_2\sin\theta}.$$

IV. HAMILTON'S PRINCIPLE FOR THE COORDINATES WITH CONSTRAINTS

Previously we argue that in order to generalize to higher-dimensional case one just make more copies of Euler equations provide that the variables are independent with each. However sometimes one needs to deal with the situation where the coordinates are not completely independent. For exampel one may have the following situation,

$$S = \int_{t_0}^{t_1} dt L(x(t), y(t), \dot{x}(t), \dot{y}, t) dt, \ \ g(x(t), y(t)) = 0.$$

$x(t)$ and $y(t)$ need to satisfy the condition $g = 0$. So when one make choice of paths with fixed start and ending points,

$$x(t_0) = x_0, \ x(t_1) = x_1, \ \ y(t_0) = y_0, \ y(t_1) = y_1.$$

We can assume that

$$x(t) = q(t) + \varepsilon f(t), \ y(t) = p(t) + \varepsilon h(t). \ \ f(t_0) = f(t_1) = g(t_0) = g(t_1) = 0.$$

x=$q(t)$ and y=$p(t)$ are supposed to be the path generating the extreme value of S. New difficulty emerges because f and h are not independent. So one cannot simply treat them as independent functions. Let's repeat the procedure from the beginning:

$$\frac{dS[\varepsilon]}{d\varepsilon}|_\varepsilon = 0.$$

Now S is still function of ε:

$$\frac{dS[\varepsilon]}{d\varepsilon}|_\varepsilon = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x}|_{\varepsilon=0} f(t) + \frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \dot{f}(t) + \frac{\partial L}{\partial y}|_{\varepsilon=0} h(t) + \frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \dot{h}(t) \right) dt$$

$$= \int_{t_0}^{t_1} dt \left[\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right) f(t) + \left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right) h(t) \right] = 0 \quad (1)$$

The main task here is to express $h(t)$ by $f(t)$ because one of them can be still free. To achieve this goal one starts from

$$g(x(t) = q(t) + \varepsilon f(t), y(t) = p(t) + \varepsilon h(t)) = 0, \Longrightarrow \frac{\partial g}{\partial x}|_{\varepsilon=0} f(t) + \frac{\partial g}{\partial y}|_{\varepsilon=0} h(t) = 0.$$

So that we have the following relation,

$$h(t) = \frac{-\frac{\partial g}{\partial x}|_{\varepsilon=0}}{\frac{\partial g}{\partial y}|_{\varepsilon=0}} f(t).$$

Insert this relation to Eq.(1) one has

$$\frac{dS[\varepsilon]}{d\varepsilon}|_{\varepsilon} = \int_{t_0}^{t_1} dt \left[\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right) f(t) + + \left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right) \frac{-\frac{\partial g}{\partial x}|_{\varepsilon=0}}{\frac{\partial g}{\partial y}|_{\varepsilon=0}} f(t) \right] = 0,$$

$$\int_{t_0}^{t_1} dt \left[\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right) + \left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right) \frac{-\frac{\partial g}{\partial x}|_{\varepsilon=0}}{\frac{\partial g}{\partial y}|_{\varepsilon=0}} \right] f(t) = 0,$$

Since $f(t)$ is still an arbitrary function such that

$$\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right) + \left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right) \frac{-\frac{\partial g}{\partial x}|_{\varepsilon=0}}{\frac{\partial g}{\partial y}|_{\varepsilon=0}} = 0.$$

One can rewrite it as follows,

$$\frac{\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right)}{\frac{\partial g}{\partial x}|_{\varepsilon=0}} = \frac{\left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right)}{\frac{\partial g}{\partial y}|_{\varepsilon=0}} = -\lambda.$$

At end we have those two equations,

$$\left(\frac{\partial L}{\partial x}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}|_{\varepsilon=0} \right) \right) + \lambda \frac{\partial g}{\partial x}|_{\varepsilon=0} = 0,$$
$$\left(\frac{\partial L}{\partial y}|_{\varepsilon=0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}|_{\varepsilon=0} \right) \right) + \lambda \frac{\partial g}{\partial y}|_{\varepsilon=0} = 0.$$

In other words, we can simple make variation on $\int (L + \lambda g) dt$ and obtain the exactly same equations! This is very convenient and useful.