Mechanics: Lagrangian Mechanics

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I. PULLEY AND THE TENSION OF THE STRING

We have learned the general method to make variation of the action with variables who are not independent. Here we practise three examples. The first example is the pulley with two masses m_1 and m_2. The Lagrangian is written as

$$L = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 - (m_1 g(-x)) - m_2 g(-y)) = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 + m_1 gx + m_2 gy.$$

Note that x and ye not independent because the length of the string connecting two masses is constant: x+y=C, $\dot{x}$=$-\dot{y}$. Therefore one can transform the whole problem into one variable case:

$$L = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}(-\dot{x})^2 + m_1 gx + m_2 g(C - x) = \frac{m_1 + m_2}{2}\dot{x}^2 + (m_1 - m_2)gx + m_2 gC.$$

The equation of motion of x is,

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g, \;\; \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}g.$$

Now we can solve the problem as the case of two variables,

$$L = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 + m_1 gx + m_2 gy + \lambda(x + y - C).$$

λ is Lagrangian multiple. Making variation on x and y we obtain the following equations,

$$m_1\ddot{x} = m_1 gx + \lambda \; m_2\ddot{y} = m_2 gy + \lambda. \; x + y = C.$$

There are three equations for three unknown variables. Now we replace y as $C - x$ in the equations,

$$m_1\ddot{x} = m_1 gx + \lambda, \;\; -m_2\ddot{x} = -m_2 gx + \lambda.$$

Subtract the second equation from the first one it is easy to obtain $\ddot{x}$ and λ which is

$$\lambda = -\frac{m_1 m_2}{m_1 + m_2} g.$$

It is easy to check that λ is indeed the tension of the string. Namely the Lagrangian multiple does correspond to the force of constraint.

II. NORMAL FORCE OF THE FALLING BODY IN INCLINE PLANE

Previously we have learned how to deal with the falling body in an accelerated incline plane. Nevertheless, we lose information of the force acting on the falling body from the incline plane. One is able to computer this quantity by the new method. Assume the variables are a the $x$coordinate of the incline plane and ζ, the horizontal coordinate between the falling body and the plane; and η the vertical one. We can easily write down the Lagrangian:

$$L = \frac{m_1}{2}\dot{a}^2 + \frac{m_2}{2}((\dot{a} + \dot{\zeta})^2 + \dot{\eta}^2) - m_2 g\eta + \lambda(\zeta \sin\theta - \eta\cos\theta).$$

The equations of motion are

$$m_1\ddot{a} + m_2(\ddot{a} + \ddot{\zeta}) = 0,$$
$$m_2(\ddot{a} + \ddot{\zeta}) = \lambda \sin\theta,$$
$$m_2\ddot{\eta} = -m_2 g - \lambda\cos\theta.$$

To solve those equations we first replace η by ζ,

$$\eta = \zeta \tan\theta.$$

The next step is insert this replacement in the third equation and express λ as function of $\ddot{\zeta}$,

$$m_2\ddot{\zeta}\tan\theta = -m_2 g - \lambda\cos\theta \Longrightarrow \lambda = -m_2\frac{\ddot{\zeta}\tan\theta + g}{\cos\theta}.$$

Sequently we can insert this expression of λ to the second equation,

$$m_2\ddot{a} + m_2\ddot{\zeta} = -m_2\tan\theta(\ddot{\zeta}\tan\theta + g) \Longrightarrow m_2\ddot{a} + m_2\sec^2\theta\ddot{\zeta} = -m_2 g\tan\theta$$
$$\Longrightarrow m_2\ddot{a}\cos^2\theta + m_2\ddot{\zeta} = -m_2 g\sin\theta\cos\theta.$$

The last step is to replace $\ddot{a}$ by $\ddot{\zeta}$ according to the first equation,

$$\ddot{a} = -\frac{m_2}{m_1 + m_2}\ddot{\zeta}.$$

Taking some time to simplify the expression,

$$\left(-\frac{m_2^2 \cos^2\theta}{m_1 + m_2} + m_2\right)\ddot{\zeta} = -m_2 g \sin\theta\cos\theta \Longrightarrow \left(\frac{m_1 + m_2\sin^2\theta}{m_1 + m_2}\right)\ddot{\zeta} = -g\sin\theta\cos\theta.$$

$$\Longrightarrow \ddot{\zeta} = \frac{-(m_1 + m_2)g\sin\theta\cos\theta}{m_1 + m_2\sin^2\theta}.$$

The minus sign is because we define the direction of motion of the incline plane is positive. Furthermore we have

$$\ddot{a} = \frac{m_2 g\sin\theta\cos\theta}{m_1 + m_2\sin^2\theta}, \quad \ddot{\eta} = \frac{-(m_1 + m_2)g\sin^2\theta}{m_1 + m_2\sin^2\theta}.$$

Naturally we have the following result.

$$\lambda = \frac{m_2 g}{\cos\theta}\left(-1 + \frac{(m_1 + m_2)\sin\theta\cos\theta\tan\theta}{m_1 + m_2\sin^2\theta}\right).$$

$$\Longrightarrow \lambda = \frac{-m_2 g}{\cos\theta}\left(\frac{(m_1 + m_2)\sin^2\theta - m_1 - m_2\sin^2\theta}{m_1 + m_2\sin^2\theta}\right).$$

$$\Longrightarrow \lambda = \frac{m_1 m_2 g\cos\theta}{m_1 + m_2\sin^2\theta}$$

One can check this result with the method in Newtonian Mechanics.

III. FALLING BALL ALONG THE BOWEL

The third example is more complicated. Assuming that there is a ball rolling along the edge of a bowel for a while and at the certain point, this ball flies away. To determine this departing point is not trivial. To analyze the motion of this ball before it flies away is as follows. The ball receives two forces, one is the gravity and the another force is the normal force from the bowel to the ball. The resultant force is centripetal force whose magnitude is determined by the velocity of the ball. In the begin the velocity of the ball is small, so the normal force must be large enough to cancel the gravity to generate the centripetal force. The gravity force projected to the tangent direction will accelerate the ball so the velocity of the ball grows. When the velocity of the ball is large enough then the gravity projected to the normal direction will be equal to centripetal force and the normal force vanishes. The point where the normal force vanishes is the position for the ball to fly away. Hence we start from the Lagrangian,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(\sqrt{x^2 + y^2} - R).$$

Here λ is the normal force. The equation of motion is given as,

$$m\ddot{x} = \lambda\frac{x}{\sqrt{x^2 + y^2}}, \quad m\ddot{y} = \lambda\frac{y}{\sqrt{x^2 + y^2}} - mg.$$

With the constraint $x^2 + y^2$=R^2 it will be most convenient to change to the following coordinate,

$$x = R\sin\theta, \quad y = R\cos\theta.$$

θ is a function of t,

$$\dot{x} = R\dot{\theta}\cos\theta, \; \dot{y} = -R\dot{\theta}\sin\theta, \; \ddot{x} = R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta, \; \ddot{y} = -R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta.$$

Now we rewrite the equations of motion as this way,

$$mR(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = \lambda\sin\theta,$$
$$mR(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) = \lambda\cos\theta - mg.$$

Multiplying the first equation with $\cos\theta$ and the second one with $\sin\theta$ then subtract them one obtains,

$$mR\ddot{\theta} = mg\sin\theta.$$

This is the equation of motion of θ. Now let us look for the angle at which the normal force vanishes,

$$\lambda = mR(\ddot{\theta}\cot\theta - \dot{\theta}^2).$$

Note that $\ddot{\theta}$ is function of t, but one can invert the function $\theta(t)$ as $t(\theta)$. Hence we can express $\ddot{\theta}$ as a function of θ. The equation of motion provides this link,

$$\lambda = 0 = mR(\frac{g}{R}\cos\theta_0 - \dot{\theta}^2(\theta = \theta_0)).$$

Now we need to express $\dot{\theta}$ as a function of θ as the case of $\ddot{\theta}$. It is actually able to be achieved by multiplying the equation of motion by the factor $\dot{\theta}$,

$$\begin{aligned} \dot{\theta}\ddot{\theta} &= \frac{g}{R}\dot{\theta}\sin\theta, \\ \frac{1}{2}\frac{d}{dt}\left(\dot{\theta}^2\right) &= -\frac{g}{R}\frac{d}{dt}\left(\cos\theta\right). \\ \Longrightarrow \quad \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2} + \frac{g}{R}\cos\theta\right) &= 0. \end{aligned}$$

Hence we know there is one quantity which keeps constant during the motion. Remember that at $t = 0$, the ball rests on the top of the bowel, so that

$$\frac{\dot{\theta}^2}{2} + \frac{g}{R}\cos\theta = constant = \frac{g}{R}(t = 0). \Longrightarrow \dot{\theta}^2 = \frac{2g}{R}(1 - \cos\theta).$$

Now we simply insert this result to the condition of $\lambda = 0$ and obtain,

$$\frac{g}{R}\cos\theta_0 - \dot{\theta}^2(\theta = \theta_0) = 0 \Longrightarrow \frac{g}{R}\cos\theta_0 = \frac{2g}{R}(1 - \cos\theta_0) \Longrightarrow \cos\theta_0 = \frac{2}{3}.$$