Mechanics: Lagrangian Mechanics

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I. GENERALIZED MOMENTUM, HAMILTONIAN AND CONSERVATION OF MECHANICAL ENERGY

One great advantage of Lagrangian Mechanics is the freedom of choosing the suitable variables. Unlike Newtonian Mechanics which is essentially a vectorial theory, Lagrangian Mechanics is a scalar theory. The action S is a scalar and the Lagrangian function is also a scalar. As long as the path giving the extreme value of the action is same, it does not matter which variables one chooses. In particular, when there are constraints, one can always reduce the number of the variables such that each variables are independent with each other. The coordinates which are not Cartesian ones are called "Generalized coordinates".

In previous section we have demonstrated that in the Lagrangian is invariant under translation then the corresponding momentum is conserved. As a matter of fact, it can be generalized to Generalized coordinate case. Assume that $L(q_1, q_2....q_k, \dot{q}_1, \dot{q}_2, ...\dot{q}_k, t)$ is the Lagrangian, furthermore we have $\frac{\partial L}{\partial q_i} = 0$. The equation becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} = 0.$$

Hence one defines the "Generalized momentum p_i=$\frac{\partial L}{\partial \dot{q}_i}$ and as a result,

$$p_i = constant \quad if \quad \frac{\partial L}{\partial q_i} = 0.$$

In other words the generalized momentum is conserved if the Lagrangian is independent of the corresponding generalized coordinate.

Moreover we know when $\frac{\partial L}{\partial t}$=0 one has

$$h = \frac{d}{dt}\left(\sum_{i=1}^{k} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0.$$

h is called hamiltonian. We have proved the case of Cartesian coordinates. It is easy to generalized to the case of generalized coordinates. Furthermore if the system is under the influence of velocity-independent conservative force then L=$T-U$. Hence we have

$$\sum_{i=1}^{k} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - T + U = constant$$

In case of Cartesian coordinate the first term is equal to $2T$ so hamiltonian will be equal to $T+U$, mechanical energy E. However in the case of generalized case it is always so. To see why one can write the kinetic energy T in term of the Cartesian coordinates first,

$$T = \sum_{a=1}^{N}\sum_{l=1}^{3} \frac{m_a}{2}\dot{r}_{al}^2.$$

To change to the generalized coordinates we start from the following relations:

$$r_{al} = r_{al}(q_1, q_2....q_k), a = 1....N, l = 1..3. \quad \dot{r}_{al} = \sum_{i=1}^{k} \frac{\partial r_a l}{\partial q_i}\dot{q}_i + \frac{\partial r_{al}}{\partial t}.$$

Therefore we have

$$T = \sum_{a,l} \frac{m_a}{2}\left(\sum_{i,j} \frac{\partial r_{al}}{\partial q_i}\frac{\partial r_{al}}{\partial q_j}\dot{q}_i\dot{q}_j + 2\sum_i \frac{\partial r_{al}}{\partial q_i}\frac{\partial r_{al}}{\partial t}\dot{q}_i + \left(\frac{\partial r_{al}}{\partial t}\right)^2\right).$$

Hence we have

$$\frac{\partial T}{\partial \dot{q}_c} = \sum_{al} m_a \sum_{i=1}^{k} \frac{\partial r_{al}}{\partial q_i}\frac{\partial r_{al}}{\partial q_c}\dot{q}_i + m_a\frac{\partial r_{al}}{\partial q_c}\frac{\partial r_{al}}{\partial t}.$$

Hence

$$\sum_{c=1}^{k} \dot{q}_c\frac{\partial T}{\partial \dot{q}_c} = \sum_c\sum_{al} m_a \sum_{i=1}^{k} \frac{\partial r_{al}}{\partial q_i}\frac{\partial r_{al}}{\partial q_c}\dot{q}_i\dot{q}_c + m_a\frac{\partial r_{al}}{\partial q_c}\frac{\partial r_{al}}{\partial t}\dot{q}_c.$$

We find that only if T is equal to the first term then $\sum_{c=1}^{k}\dot{q}_c\frac{\partial T}{\partial \dot{q}_c}$=2T. In more formal language we will say it is true only when T is degree-2 homogenous polynomial of $\dot{q}_i$. A homogenous polynomial of degree N means the polynomial is able to written as

$$P(q_1, q_2.....q_k) = \sum C_{a_1,a_2,....a_k} q_1^{a1} q_2^{a2}.....q_k^{a_k}, \quad a_1 + a_2 + a_3 + ...a_k = N.$$

It is left for exercise to show that

$$\sum_{i=1}^{k} q_i\frac{\partial P}{\partial q_i} = NP.$$

So if T is degree-two polynomial of $\dot{q}_i$ surely h=E.

II. MECHANICAL SIMILARITY

If the potential has the following scaling behaviour,

$$U(r_i) \to U(\lambda r_i) = \lambda^{\alpha} U(r_i).$$

Then we could ask whether the system can be the same if we have made such a transform $\vec{r} \longrightarrow \lambda\vec{r}$. This question is particular transparent in Lagrangian Mechanics. As long as L also changes to $\lambda^{\alpha} L$ then the action is just multiplied by a factor, the physics will be the same. Hence one just needs point out the situation in which T is also transformed to $\lambda^{\alpha} T$. One can assume the following transform $t \longrightarrow \beta t$ to do the trick. Under these two transforms we notice that $\frac{dx}{dt} \longrightarrow \frac{\lambda}{\beta}\frac{dx}{dt}$. As a result we have

$$T \longrightarrow \left(\frac{\lambda}{\beta}\right)^2 T.$$

To make T is scaled by the factor λ^{α} as U does, we find that β=$\lambda^{1-\frac{\alpha}{2}}$. Therefore the system is invariant if

$$\vec{r} \longrightarrow \lambda\vec{r}, t \longrightarrow \lambda^{1-\frac{\alpha}{2}} t, \;\; U(\vec{r}) \longrightarrow \lambda^{\alpha} U(\vec{r}),$$

Under such transform one also has

$$\vec{v} \longrightarrow \lambda^{\alpha/2}\vec{v}, \;\; E \longrightarrow \lambda^{\alpha} E, \;\; \vec{L} = \vec{r} \times \vec{p} \longrightarrow \lambda^{1+\alpha/2}\vec{L}.$$

One particular application of "Mechanical similarity" is Kepler's third law. For any planet in the solar system, its period τ and its orbit radius R satisfies the following relation,

$$\frac{\tau^2}{R^3} = constant.$$

It is easy to see this particular combination $\frac{\tau^2}{R^3}$ transforms as $\lambda^{-3+2-\alpha}\frac{\tau^2}{R^3}$. Hence one concludes that $\alpha = -1$. Indeed the potential of gravity is proportional to r^{-1} because the gravity is inverse-squared force.

III. VIRIAL THEOREM

If the potential is scaled as power law, then we can derive something very useful for the periodic motion, called virial theorem. WE start from the following identity

$$2T = \sum_a \vec{p}_a \cdot \vec{v}_a = \frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right) - \sum_a \vec{r}_a \cdot \dot{\vec{p}}_a.$$

For a periodic motion, one can defined the periodic average of a physical observable $f(t)$ as

$$\bar{f} = \frac{1}{\tau}\int_0^\tau f(t)dt.$$

Any total derivative term give zero periodic average so one has

$$2\bar{T} = \sum_a \overline{\vec{r}_a \cdot \dot{\vec{p}}_a} = \sum_a \overline{\vec{\ }_a \cdot \nabla_a U}.$$

If potential energy is a homogenous function of degree of k in the radius vector r_a. Then we have

$$2\bar{T} = k\bar{U}.$$

Since $T + U$=E, one has

$$\bar{U} = \frac{2}{k+2}E, \;\; \bar{T} = \frac{k}{k+2}E.$$

When k=-1, such as in the case of gravity, then $\bar{U} = 2E$ and $\bar{T} = -E$, here $E < 0$. When $k = -2$ then one notice that E must be zero. It sounds strange. However, remember virial theorem is only applicable in the periodic motion. Therefore it just means when the force is proportional to r^{-3}, only when $E = 0$ the orbit is closed.