Mechanics: Lagrangian Mechanics

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I. 1-DIMENSIONAL MOTION

In one-dimensional system, if the potential is independent of velocity and time, then

$$L = \frac{m}{2}\dot{x}^2 - U(x).$$

The equation of motion is

$$m\ddot{x} = -\frac{dU}{dx}.$$

Multiplying $\dot{x}$ at the both side we have

$$m\dot{x}\ddot{x} = -\frac{dU}{dx}\frac{dx}{dt}, \Longrightarrow \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right) = -\frac{dU}{dt}, \Longrightarrow \frac{m}{2}\dot{x}^2 + U = constant = E.$$

It is conservation of mechanical energy. The value of E is given by the initial position $x(t=0)$ and the initial velocity $\dot{x}(t=0)$. If one know the value of E and $U(x)$, then one can solve the equation of motion by integration. First, the speed can be written as function of $U(x)$ and E

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\sqrt{E - U(x)}.$$

By directly integration one solves the equation,

$$t(x) = \sqrt{\frac{m}{2}}\int_{x=x(t=0)}^{x} \frac{dx'}{\sqrt{E - U(x')}}.$$

Sometimes the particle is trapped inside the potential. For example when $x_a \leq x \leq x_b$, $E \geq U(x)$, otherwise $E \leq U(x)$. Then the particle can only move between x_a and x_b. Why? because we notice the kinetic energy T must be non-negative, therefore E cannot be smaller than $U(x)$. The points x_a and x_b is called "turning point". There the velocity $\dot{x}(x_a)$=$\dot{x}(x_b)$=0 and one finds that the particle cannot continue to go ahead because the potential is too high. Therefore the particle must turn around. This is called "Barrier of

Potential". So what is the period of the motion between two turning points? Note that the time from the particle from x_a to x_b must be the same as one from x_b to x_a because the whole system is invariant under time reversal $t \longrightarrow -t$. One can imagine to take video of the motion from x_a to x_b then plays it back, it will agree with the actual motion from x_b to x_a. Hence the period of the whole motion is

$$T(E) = \sqrt{2m}\int_{x_a}^{x_b} \frac{dx}{\sqrt{E-U(x)}}.$$

II. PERIOD OF OSCILLATION IN THE TRAP AND THE SHAPE OF POTENTIAL

One interesting question is to invert $T(E)$ to acquire the potential $U(x)$. To do so we take the "resting point x_c as the origin. The point x_c is defined as $\frac{dU(x_c)}{dx}$=0. The right-handed side position is denoted as x_2 and the left-handed position is denoted as x_1. It is clear that there is one-to-one corresponding between U and x_2 , also there is similar one-to-one corresponding between x_1 and U. So instead x, we can change variable from x to U and the period of the motion can be expressed as

$$\begin{aligned} T(E) &= \sqrt{2m}\int_0^E \frac{dx_2}{dU}\frac{dU}{\sqrt{E-U}} + \sqrt{2m}\int_E^0 \frac{dx_1}{dU}\frac{dU}{\sqrt{E-U}} \\ &= \sqrt{2m}\int_0^E dU\left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right]\frac{dU}{\sqrt{E-U}}. \end{aligned}$$

The crucial step is the following one. Let us to make such an integration,

$$\int_0^\alpha \frac{T(E)}{\sqrt{\alpha-E}} = \sqrt{2m}\int_0^\alpha dU \int_0^E dE\left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right]\frac{1}{\sqrt{(\alpha-E)(E-U)}}.$$

By change the order of integration:

$$\int_0^\alpha dU\int_0^E dE = \int_0^\alpha dU\int_U^\alpha dE.$$

The integration becomes

$$\int_0^\alpha \frac{T(E)}{\sqrt{\alpha-E}} = \sqrt{2m}\int_0^\alpha dU \int_U^\alpha dE\left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right]\frac{1}{\sqrt{(\alpha-E)(E-U)}}.$$

One needs evaluate the following integral which is actually π.

$$\int_U^\alpha \frac{dE}{\sqrt{(\alpha - E)(E_U)}} = \int_0^1 \frac{dz}{\sqrt{z(1-z)}}, \quad E = U + z(\alpha - U).$$
$$= \int_{-\pi/2}^{\pi/2} \frac{\frac{1}{2}\cos\theta d\theta}{\frac{1}{2}\cos\theta} = \pi, \quad z = \frac{1}{2} + \frac{1}{2}\sin\theta.$$

As a result one reaches

$$\int_0^\alpha \frac{T(E)dE}{\sqrt{\alpha - E}} = \pi\sqrt{2m}[x_2(\alpha) - x_1(\alpha)].$$

Put the value of α to be U one has

$$\Delta x(U) = x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U dE \frac{T(E)}{\sqrt{U - E}}.$$

One particular interesting case is to see what kind of potential to generate $T(E)$ as constant. The answer is $U(x)$=$k(x - x_0)^2$. Here k is a constant.