Mechanics: Lagrangian Mechanics

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I. 2-DIMENSIONAL MOTION OF CENTRAL FORCE FIELD

Here we deal with the problem of the motion of two-body system with the central force. The Lagrangian of such a system can be written as

$$L = \frac{m_1}{2}\dot{\vec{r}}_1^2 + \frac{m_2}{2}\dot{\vec{r}}_2^2 - U(|\vec{r}_1 - \vec{r}_2|).$$

This is a system of six degrees of freedom. U only depends on $|\vec{r}_1 - \vec{r}_2|$ is because of the symmetry, Galilean invariance and rotational invariance. Previously mentioned that one can apply this coordinates:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2.$$

and transform the Lagrangian into

$$L = \frac{m_1 + m_2}{2}\dot{\vec{R}}^2 + \frac{\mu}{2}\dot{\vec{r}}^2 - U(|\vec{r}|).$$

The Lagrangian of $\vec{R}$ and $\vec{r}$ is separated. It is easy to see the motion of $\vec{R}$ is trivial. Hence we choose our origin to be the position of $\vec{R}$. In other words, we just focus on the motion of $\vec{r}$. So far we have reduced this six-dimensional problem into a three-dimensional problem.

Furthermore we know there is the rotational invariance for the Lagrangian: $L{=}\frac{\mu}{2}\dot{\vec{r}}^2 - U(|\vec{r}|)$. We know the angular momentum is conserved, that is

$$\vec{r} \times \mu\dot{\vec{r}} = \vec{l} = constant.$$

We then choose $\hat{L}$ the z axis direction. Since $\vec{r}$ and $\vec{v}$ are normal to $\vec{L}$, we know the motion occurs in the x-y plane. Here we choose the polar coordinate:

$$L = \frac{\mu}{2}\dot{\vec{r}}^2 - U(|\vec{r}|).$$

One must be aware that when one use the polar coordinate, the translation invariance is lost. It is because the origin is unique point in the polar coordinate. In this problem, we choose the origin to be the position of the centre mass, therefore it is fine to use the polar coordinate. The 2-D Lagrangian now looks like,

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r).$$

The equations of motion are

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{dU}{dr}, \quad \frac{d}{dt}(\mu r^2\dot{\theta}) = 0.$$

The second one tells us that the conservation of angular momentum,

$$\mu r^2\dot{\theta} = constant = l.$$

The first one is equation of r. As a matter of fact, it also implies one conserved quantity. Multiplying $\dot{r}$,

$$\mu\dot{r}\ddot{r} = \mu r\dot{r}\dot{\theta}^2 - \frac{dU}{dr}\dot{r}, \longrightarrow \mu\dot{r}\ddot{r} = \frac{l^2}{\mu r^3}\dot{r} - \frac{dU}{dr}\dot{r},$$
$$\longrightarrow \frac{d}{dt}\left(\frac{\mu}{2}\dot{r}^2\right) = \frac{d}{dt}\left(\frac{-l^2}{2\mu r^2}\right) - \frac{dU}{dt}, \longrightarrow \frac{d}{dt}\left(\frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)\right) = 0.$$

What is this conserved quantity? Remember here we have $\frac{\partial L}{\partial t}$=0, $L = T - U$ and T is degree-2 homogenous polynomial of $\dot{r}$ and $\dot{\theta}$. Therefore we know the mechanical energy is conserved. It is not difficult to show that indeed the conserved quantity is the mechanical energy E,

$$\frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = E.$$

It is similar to the case of 1-dimensional motion we deal in the previous section. Namely we have

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}.$$

One can integrate $t(r)$. However this is a 2-D motion, we are more interested in the orbit of the motion, namely $r(\theta)$. To obtain the equation of the orbit we can apply the conservation of momentum and have,

$$dt = d\theta\frac{dt}{d\theta} = \frac{d\theta}{\dot{\theta}} = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}.$$

Hence we reach,

$$d\theta = \frac{\dot{\theta} dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}} = \frac{\frac{l}{\mu r^2} dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}$$

At end we can calculate the orbit of any central force potential $U(r)$,

$$\theta = \int \frac{\frac{l}{\mu r^2} dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}.$$

II. KEPLER'S PROBLEM

Historic importance of the Kepler's problem cannot be emphasized. When $U(r){=}-\frac{\alpha}{r}$, what is the orbit? The answer is actually dependent on the value of E and l. Since it is a 2-dimensional motion, therefore the orbit is completely determined by these two conserved quantities. One can apply the technique learned from the last section with a tiny modification. We need change the potential U by the effective potential $U_{eff}{=}U(r) + \frac{l^2}{2\mu r^2}$. One can easy to see when $\frac{dU}{dr}(r_0){=}0$ with $r_0{=}\frac{l^2}{\alpha\mu}$ with $E{=}U_{eff}(r_0){=}-\frac{\alpha^2\mu}{2l^2}$. This is the minimum value of E. When $E_{min} \leq E \leq 0$, what is the orbit? To solve this problem we just need evaluate the following integral,

$$\begin{aligned}
\theta &= \int \frac{\frac{l}{\mu r^2} dr}{\sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{l^2}{2\mu r^2}\right)}} = \frac{l}{\sqrt{2\mu}} \int \frac{\frac{dr}{r^2}}{\sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{l^2}{2\mu r^2}\right)}} \\
&= \frac{l}{\sqrt{2\mu}} \int \frac{-du}{\sqrt{E + \alpha u - \frac{l^2}{2\mu} u^2}} = \frac{-l}{\sqrt{2\mu}} \int \frac{du}{\sqrt{E + \frac{\mu\alpha^2}{2l^2} - \frac{l^2}{2\mu}\left[u - \frac{\mu\alpha}{l^2}\right]^2}} \\
&= \frac{-l}{\sqrt{2\mu}} \int \frac{dw}{\sqrt{E + \frac{\mu\alpha^2}{2l^2} - \frac{l^2}{2\mu} w^2}} = \frac{l}{\sqrt{2\mu}} \sqrt{\frac{2\mu}{l^2}} \cos^{-1}\left(\sqrt{\frac{\frac{l^2}{2\mu}}{E + \frac{\mu\alpha^2}{2l^2}}} w\right) \\
&= \cos^{-1}\left(\sqrt{\frac{\frac{l^2}{2\mu}}{E + \frac{\mu\alpha^2}{2l^2}}} w\right).
\end{aligned}$$

$$\begin{aligned}
\cos\theta &= \sqrt{\frac{\frac{l^2}{2\mu}}{E + \frac{\mu\alpha^2}{2l^2}}} \left(\frac{1}{r} - \frac{\mu\alpha}{l^2}\right) \Longrightarrow \frac{1}{r} = \sqrt{\frac{E + \frac{\mu\alpha^2}{2l^2}}{\frac{l^2}{2\mu}}} \cos\theta + \frac{\mu\alpha}{l^2} \\
&\Longrightarrow \frac{l^2/\mu\alpha}{r} = 1 + \sqrt{1 + \frac{2El^2}{\alpha^2\mu}} \cos\theta.
\end{aligned}$$

This means the orbit is an ellipse.