Mechanics: Lagrangian Mechanics

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(Dated: October 19, 2012)

This lecture was given on Oct 18th.

I. KEPLER'S PROBLEM

The result of the integration is

$$\frac{K}{r} = 1 + \epsilon \cos\theta.$$

Here we find that

$$K = \frac{l^2}{\mu\alpha}, \ \ \epsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha^2}}.$$

From the equation of the orbit we know that

$$r_{min} = \frac{K}{1+\epsilon}, \ r_{Max} = \frac{K}{1-\epsilon}.$$

Using the Cartesian coordinate the equation of the orbit becomes

$$\frac{\left(x + \frac{\epsilon K}{1-\epsilon^2}\right)^2}{\left(\frac{K}{1-\epsilon^2}\right)^2} + \frac{y^2}{\left(\frac{K}{\sqrt{1-\epsilon^2}}\right)^2} = 1.$$

Compared with the standard from of ellipse curve one knows,

$$a = \frac{K}{1-\epsilon^2} = \frac{\alpha}{-2E} = \frac{r_{min} + r_{Max}}{2}, \ \ b = \frac{K}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{-2\mu E}}.$$

Now we can explicitly demonstrate Kepler's third law. The period of planet surrounds sun is τ:

$$\begin{aligned} A &= \pi ab, \frac{dA}{dt} = \frac{l}{2\mu}, \longrightarrow \tau = \frac{\pi ab}{\frac{dA}{dt}} = \pi\alpha\sqrt{\frac{\mu}{2|E|^3}}. \\ a^3 &= \frac{\alpha^3}{8|E|^3}, \ T^2 = \frac{\pi^2\alpha^2\mu}{2|E|^3}, \ \longrightarrow \frac{T^2}{a^3} = \frac{4\pi^2\mu}{\alpha}. \end{aligned}$$

$$\frac{4\pi^2\mu}{\alpha} = 4\pi^2\frac{m_1 m_2}{(m_1+m_2)Gm_1m_2} = \frac{4\pi^2}{m_1+m_2} \sim \frac{4\pi^2}{m_1}.$$

Since m_1 is the mass of sun and m_2 is planet's mass which is much smaller than m_1. In astronomy we also want to know $r(t)$ and $\theta(t)$. However the integrals turn out to be not elementary functions. Nevertheless there is an alternative way. The integral as follows,

$$
\begin{aligned}
t &= \int \frac{dr}{\sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{l^2}{2\mu r^2}\right)}} \\
&= \sqrt{\frac{\mu}{-2E}} \int \frac{rdr}{\sqrt{-r^2 + \frac{\alpha}{|E|} r - \frac{l^2}{2\mu |E|}}} \\
&= \sqrt{\frac{\mu a}{\alpha}} \int \frac{rdr}{\sqrt{a^2\epsilon^2 - (r-a)^2}}
\end{aligned}
$$

Making the substitution as follows,

$$r - a = -a\epsilon\cos\xi.$$

Then we have

$$r = a(1 - \epsilon\cos\xi),\ t = \sqrt{\frac{\mu a^3}{\alpha}}(\xi - \epsilon\sin\xi) + constant.$$

One can verify the angle ξ actually is the angle between the line between the planet and the central of the orbit and the major axis.

II. PERIHELION PRECESSION

The orbit of a two-body system with central force with two turning points is not necessary periodic as one-dimensional case. Because the turning points x_A and x_B just tell you $\dot{r}$=0 and r=R_{min} or r_{Max}, however after one "period" the angle θ is not necessarily return the initial angle θ_0. Therefore one can calculate the difference of the angles of turning points:

$$\Delta\theta = 2\int_{r_{min}}^{r_{Max}} \frac{\frac{ldr}{\mu r^2}}{\sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{l^2}{2\mu r^2}\right)}}.$$

The orbit will be close only if

$$\Delta\theta = \frac{2m\pi}{n}.$$

In general the planet's orbit is not close because the gravity potential is not simply $\frac{\alpha}{r}$ once the attraction from other planets included. Human has observed this phenomenon for long time. The formula of $\Delta\theta$ can be changed into

$$\Delta\theta = -2\frac{\partial}{\partial l}\int_{r_{min}}^{r_{Max}} \sqrt{\left[2\mu(E-U(r)) - \frac{l^2}{r^2}\right]}dr.$$

Now the potential is the combination of the gravitation potential between the planet and the sun and other much small potential denoted as $\beta\delta U(r)$. Here $\beta << 1$ is a parameter,

$$U(r) = -\frac{\alpha}{r} + \beta\delta U(r).$$

Then we can make Taylor expansion over β

$$\sqrt{2\mu(E-U(r)) - \frac{l^2}{r^2}} = \sqrt{2\mu\left(E+\frac{\alpha}{r}\right) - \frac{l^2}{r^2}} - \frac{2\mu\beta\delta U(r)}{\sqrt{2\mu\left(E+\frac{\alpha}{r}\right) - \frac{l^2}{r^2}}} + \mathcal{O}(\beta^2).$$

If we are only interested in the effect up to $\mathcal{O}(\beta)$ then one can adopt the following approximation,

$$\Delta\theta = \frac{\partial}{\partial l}\int_{r_{min}}^{r_{Max}} \frac{2\mu\beta\delta U(r)dr}{\sqrt{2\mu\left(E+\frac{\alpha}{r}\right) - \frac{l^2}{r^2}}}.$$

The next step is to change the variable from r to θ,

$$dr = \frac{dt}{\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}} = \frac{\frac{\mu r^2}{l}d\theta}{\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}} \sim \frac{\frac{\mu r^2}{l}d\theta}{\sqrt{\frac{2}{\mu}\left(E+\frac{\alpha}{r}-\frac{l^2}{2\mu r^2}\right)}}.$$

Hence

$$\frac{dr}{\sqrt{2\mu\left(E+\frac{\alpha}{r}\right) - \frac{l^2}{r^2}}} = \frac{\frac{\mu}{l}r^2 d\theta}{\mu\sqrt{\frac{2}{\mu}\left(E+\frac{\alpha}{r}\right) - \frac{l^2}{\mu^2 r^2}}} = \frac{1}{l}\frac{r^2 d\theta}{\sqrt{\frac{2}{\mu}\left(E+\frac{\alpha}{r}-\frac{l^2}{2\mu r^2}\right)}}.$$

The formula of $\Delta\theta$ becomes

$$\Delta\theta \sim \frac{\partial}{\partial l}\left(\frac{2\mu}{l}\int_0^{\pi} r^2\beta\delta U(r(\theta))d\theta\right).$$

If $\delta U{=}\frac{1}{r^2}$ then one can easily know $\Delta\theta{=}{-}\frac{2\pi\beta}{l^2}{=}{-}\frac{2\pi\beta}{a(1-\epsilon^2)\alpha}$.

Question: Please find that the $\Delta\theta$ in the cases of (a) $\delta U{=}\frac{1}{r^3}$, (b) $\delta U{=}\frac{1}{r^4}$.