Mechanics: Lagrangian Mechanics

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I. ELASTIC COLLISION

The phenomenon of collision is very subject in Mechanics. From Galilean invariance we know the interacting force between two bodies are of the same size and opposite direction. Hence the momentum of centre of mass is conserved. The elastic collision is a collision in which the kinetic energy and the momentum centre of mass are both conserved. Whether a collision is elastic or not is actually frame-independent, namely if one collision is verified to be elastic in one frame, then after Galilean transform, this collision is still elastic one in the new frame. That is, if $T_1 + T_2 - T_1' - T_2'$=0 in one frame then it is zero after any Galilean transform:

$$\begin{aligned}
& \Delta T_0 = T_1 + T_2 - T_1' - T_2' = \frac{m_1}{2}|\vec{v}_1|^2 + \frac{m_2}{2}|\vec{v}_2|^2 - \frac{m_1}{2}|\vec{v'}_1|^2 - \frac{m_1}{2}|\vec{v'}_2|^2 = 0 \\
\Longrightarrow & \Delta T = T_1 + T_2 - T_1' - T_2' = \frac{m_1}{2}|\vec{u}_1|^2 + \frac{m_2}{2}|\vec{u}_2|^2 - \frac{m_1}{2}|\vec{u'}_1|^2 - \frac{m_1}{2}|\vec{u'}_2|^2 = 0 \\
= & \frac{m_1}{2}|\vec{v}_1 + \vec{V}_R|^2 + \frac{m_2}{2}|\vec{v}_2 + \vec{V}_R|^2 - \frac{m_1}{2}|\vec{v'}_1 + \vec{V}_R|^2 - \frac{m_1}{2}|\vec{v'}_2 + \vec{V}_R|^2 = 0 \\
= & \Delta T_0 + (m_1\vec{v}_1 + m_2\vec{v}_2 - m_1\vec{v'}_1 - m_2\vec{v'}_2)\cdot\vec{V}_R \\
= & \Delta T_0 + (m_1 + m_2)(\vec{v}_c - \vec{v'}_c)\cdot\vec{V}_R = \Delta T_0.
\end{aligned}$$

If we choose the centre of mass frame where the momentum of the centre of mass is zero, then it can be shown that the size of the momentum of each particle is the same, only the direction is changed. Because in the C.M frame the initial momentum of two particles are $\vec{p}_1$ and $\vec{p}_2$=$-\vec{p}_1$. The final momentum of each particle is denoted as $\vec{p'}_1$ and $\vec{p'}_2$=-$\vec{p'}_2$. Then $|\vec{p}_1|$=$|\vec{p'}_1|$, $|\vec{p}_2|$=$|\vec{p'}_2|$. The proof is simple,

$$T_1 + T_2 = \frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2} = T_1' + T_2' = \frac{|\vec{p'}_1|^2}{2m_1} + \frac{|\vec{p'}_2|^2}{2m_2}.$$

$$\frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2} = \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)|\vec{p}_1|^2 = \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)|\vec{p'}_1|^2, \longrightarrow |\vec{p}_1| = |\vec{p'}_1|$$

We define the angle between the initial and final momentum of the first particle as θ. Surely we have $\cos\theta = \hat{p}_1 \cdot \hat{p'}_1$. It is trivial to see the angle between the initial and final momentum of the second particle is also θ.

Note that there are two quantities are Galilean invariant, those are $\vec{v} = \vec{v}_1 - \vec{v}_2$ and $\vec{v'} = \vec{v'}_1 - \vec{v'}_2$. Also we know $|\vec{v}| = |\vec{v'}$=v since

$$|\vec{v}| = |\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}| = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)|\vec{p}_1|, \quad |\vec{v'}| = |\frac{\vec{p'}_1}{m_1} - \frac{\vec{p'}_2}{m_2}| = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)|\vec{p'}_1|.$$

We can express the final velocities $\vec{v'}_1$ and $\vec{v'}_2$ as follows,

$$\vec{v'}_1 = \frac{m_2 v}{m_1 + m_2}\hat{n}, \quad \vec{v'}_2 = -\frac{m_1 v}{m_1 + m_2}\hat{n}.$$

This is a very simple expression but it only is true in the C.M frame.

II. LAB FRAME AND CM FRAME

In the previous section we have very simple formula for the final velocities. Now we can make Galilean transform this result to arbitrary frame,

$$\vec{u'}_1 = \frac{m_2 v}{m_1 + m_2}\hat{n} + \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}, \quad \vec{u'}_2 = -\frac{m_1 v}{m_1 + m_2}\hat{n} + \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}.$$

Or we can written in term of the momentum:

$$\vec{p'}_1 = \frac{m_1 m_2 v}{m_1 + m_2}\hat{n} + \frac{m_1}{m_1 + m_2}\left(\vec{p}_1 + \vec{p}_2\right), \quad \vec{u'}_2 = -\frac{m_1 m_2 v}{m_1 + m_2}\hat{n} + \frac{m_2}{m_1 + m_2}\left(\vec{p}_1 + \vec{p}_2\right)$$

We can assign $\vec{OC} = \frac{m_1 m_2 v}{m_1 + m_2}\hat{n}$, $\vec{AO} = \frac{m_1}{m_1 + m_2}\left(\vec{p}_1 + \vec{p}_2\right)$ $\vec{OB} = \frac{m_2}{m_1 + m_2}\left(\vec{p}_1 + \vec{p}_2\right) = \left(\frac{m_2}{m_1}\right)\vec{AO}$. Hence $\vec{AC} = \vec{p'}_1$ and $\vec{CB} = \vec{p'}_2$. This provides a very simply way to determine $\vec{p'}_1$ and $\vec{p'}_2$.

The frame where $\vec{u}_2 = 0$ is particularly interesting because most of scattering experiments have been carried out there. It is called "Lab" frame. It is clear that $\vec{OC} = \vec{OB}$ in this frame. So one can draw a circle and the origin locates at O, B and C are on the circle. Since $|\vec{AO}| = \left(\frac{m_1}{m_2}\right)|\vec{OB}|$. When $m_1 < m_2$ the point A is in the circle. On the other hand, when $m_1 > m_2$ the point A is out the circle. So what is the angle between the final and initial momentum of each particle? We define $\cos\psi = \hat{p}_1 \cdot \hat{p'}_1$ and e $\cos\chi = \hat{p}_2 \cdot \hat{p'}_2$. The angle ψ is the angle between $\vec{AC}$ and $\vec{AO}$. There exists a point D between A and B and $\vec{CD} \perp \vec{AB}$. Assume $|\vec{OB}| = \rho$ then one obtains

$$\tan\psi = \frac{|\vec{CD}|}{|\vec{AO} + \vec{OD}|} = \frac{\rho\sin\theta}{\frac{m_1}{m_2}\rho + \rho\cos\theta} = \frac{m_2\sin\theta}{m_1 + m_2\cos\theta}. \ \chi = \frac{\pi - \theta}{2}.$$

The magnitude of the final momentum of the first particle in the Lab frame is given by

$$|\vec{p'}_1| = |\vec{AO} + \vec{OC}| = \rho\sqrt{1 + \left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right)\cos\theta} = \frac{m_1 v}{m_1 + m_2}\sqrt{m_2^2 + m_1^2 + 2m_1 m_2 \cos\theta}.$$

In term of the final velocity,

$$|\vec{u'}_1| = \frac{\sqrt{m_2^2 + m_1^2 + 2m_1 m_2 \cos\theta}}{m_1 + m_2} v.$$

The final momentum of the second particle is given as

$$|\vec{p'}_2| = 2\rho \sin\frac{\chi}{2} = \frac{2m_1 m_2}{m_1 + m_2} v \sin\frac{\chi}{2}.$$

The final velocity of the second particle is

$$|\vec{u'}_2| = \frac{2m_1}{m_1 + m_2} v \sin\frac{\chi}{2}.$$

When $m_1 > m_2$ one can find that there is ψ_{Max}. It is easy to see

$$\psi_{Max} = \sin^{-1}\frac{|\vec{OC}|}{|\vec{OA}|} = \sin^{-1}\frac{m_2}{m_1}.$$

For the special case $m_1 = m_2$ we have

$$\psi = \frac{\theta}{2},\ \chi = \frac{\pi - \theta}{2},\ \psi + \chi = \frac{\pi}{2}.$$