Mechanics: Lagrangian Mechanics

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I. SCATTERING

Scattering is an important method to study the property of the material in general. In particle, in particle physics scattering is almost the only way to learn the interaction between elementary particles. Its importance cannot be overemphasized.

Let us deal with the scattering process in the C.M frame. We take the C.M as the force center. The scattering between two particles now can be treated as a projectile with reduced mass $\mu=\frac{m_1 m_2}{m_1+m_2}$ to interact with the force center. The impact parameter ρ is defined as the perpendicular distance between the projectile and the centre of force initially. When ρ is kept the same, the projectile will be scattered to certain angle χ. We define the cross section as the following way. n is the incident flux of the projectile in unit time for unit area. From ρ to $\rho + d\rho$ there are dN projectiles pass through per unit time. That is

$$dN = ndA = n2\pi\rho d\rho.$$

We then define $d\sigma$ as dN/n and expree it as a function of χ and therefore a function of $d\Omega$,

$$d\sigma = dN/n = 2\pi\rho d\rho = 2\pi\rho|\frac{d\rho}{d\chi}|d\chi = \frac{\rho}{\sin\chi}|\frac{d\rho}{d\chi}|d\Omega.$$

The scattered projectiles are scattered between Ω and $\Omega + d\Omega$. The scattering process can be taken as an elastic collision at initial and final states when the potential is zero. The total energy of the project is denoted as E and its anglue momentum respect with the centre of the mass is l. Then we have

$$E = \frac{1}{2}\mu v_\infty^2, \;\; l = \mu\rho v_\infty.$$

Applying the theory of orbit we just have learned immediately we have

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{l/r^2 dr}{\sqrt{2\mu(E-U(r)) - l^2/r^2}} = \int_{r_{min}}^{\infty} \frac{\rho/r^2 dr}{\sqrt{1-\rho^2/r^2 - 2U(r)/\mu v_\infty^2}}.$$

We notice that $\chi{+}2\phi_0{=}\pi$, such that one can derive the cross section from the above equation. In other words one can derive the cross section as long as we know the potential form $U(r)$.

II. RUTHERFORD FORMULA

If $U(r){=}\frac{\alpha}{r}$, and set $u{=}\frac{1}{r}$:

$$\begin{aligned}
\phi_0 &= \int_{r_{min}}^{\infty} \frac{\rho/r^2 dr}{\sqrt{1-\rho^2/r^2-2\alpha/r\mu v_\infty^2}} = \int_{u_{Max}}^{0} \frac{-\rho du}{\sqrt{1-\rho^2 u^2 - 2\left(\alpha/\mu v_\infty^2\right) u}} \\
&= \int_{0}^{u_{Max}} \frac{\rho du}{\sqrt{1+\frac{\alpha^2}{\mu^2 v_\infty^4 \rho^2} - \rho^2\left(u+\frac{\alpha}{\mu v_\infty^2 \rho^2}\right)^2}} \\
&= \int_{-\frac{\alpha}{\mu v_\infty \rho^2}}^{w_{Max}} \frac{\rho dw}{\sqrt{1+\frac{\alpha^2}{\mu^2 v_\infty^4 \rho^2} - \rho^2 w^2}} = \cos^{-1}\left(\left[\frac{\rho}{\sqrt{1+\frac{\alpha^2}{\mu^2 v_\infty^4 \rho^2}}}\left(\frac{1}{r}+\frac{\alpha}{\mu v_\infty^2 \rho^2}\right)\right]\right)\Big|_{r=r_{min}}^{r=\infty}.
\end{aligned}$$

Here $w{=}\frac{1}{\rho}\left(1+\frac{\alpha^2}{\mu^2 v_\infty^4 \rho^2}\right)^{1/2}\cos\theta$. Then we have

$$\frac{l^2}{2\mu r_{min}^2}+\frac{\alpha}{r_{min}} = E \longrightarrow r_{min} = \frac{\alpha+\sqrt{\alpha+\mu^2\rho^2 v_\infty^4}}{\mu v_\infty^2}.$$

$\frac{1}{r_{min}}{=}\frac{-\alpha}{\mu\rho^2 v_\infty^2}+\frac{1}{\rho}\sqrt{1+\frac{\alpha^2}{\rho^2\mu^2 v_\infty^4}}$.

Therefore we have

$$\phi_0 = \cos^{-1}\frac{\alpha/(\mu v_\infty^2 \rho)}{\sqrt{1+(\alpha/(\mu v_\infty^2\rho))^2}} - \cos^{-1}(1) = \cos^{-1}\frac{\alpha/(\mu v_\infty^2 \rho)}{\sqrt{1+(\alpha/(\mu v_\infty^2\rho))^2}}.$$

At the end we reach the following relation,

$$\cos\phi_0 = \frac{\alpha/(\mu v_\infty^2 \rho)}{\sqrt{1+(\alpha/(\mu v_\infty^2\rho))^2}} \longrightarrow \tan^2\phi_0 = \rho^2\left(\frac{\mu^2 v_\infty^4}{\alpha^2}\right).$$

Remember $\chi+2\phi_0{=}\pi$ so that

$$\rho^2 = \left(\frac{\alpha^2}{\mu^2 v_\infty^4}\right)\cot^2\frac{\chi}{2}.$$

It is then straightforward to reach the following result,

$$d\sigma = \pi\left(\frac{\alpha^2}{\mu^2 v_\infty^4}\right)\frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}d\chi = \left(\frac{\alpha^2}{\mu^2 v_\infty^4}\right)\frac{d\Omega}{\sin^4\frac{\chi}{2}}.$$

It is called Rutherford formula. Historically it is very famous because it is the formula Rutherford used to demonstrate the positive charge of the atom is located in a centre.

III. HARD-SPHERE SCATTERING

The meaning of the cross section is most easy to catch in the case of the hard-sphere scattering. Hard-sphere potential means there is a ball surrounding the centre of the force which is not penetrable. In other words, we can assume $U(r)$=∞ for r¿a and $U(r)$=0 for $r \leq R$. Then set r=$\rho/\sin\theta$,

$$\begin{aligned}\phi_0 &= \int_{r_{min}}^{\infty} \frac{\rho/r^2 dr}{\sqrt{1-\rho^2/r^2-2U(r)/\mu v_\infty^2}}. = \int_a^\infty \frac{\rho/r^2 dr}{\sqrt{1-\rho^2/r^2}} \\ &= \int_{\sin^{-1}\left(\frac{\rho}{a}\right)}^{0} \frac{-\sin^2\theta\cot\theta\csc\theta d\theta}{\cos\theta} = -\int_{\sin^{-1}\left(\frac{\rho}{a}\right)}^{0} d\theta = \sin\left(\frac{\rho}{a}\right)\end{aligned}$$

Note that when $\rho > a$ then there is no scattering, the projectile will just pass by. Therefore we have $\sin\phi_0$=$\frac{\rho}{a}$. Such that ρ=$a\sin\phi_0$=$a\sin\left(\frac{\pi}{2}-\frac{\chi}{2}\right)$=$a\cos\left(\frac{\chi}{2}\right)$.

$$|\frac{d\rho}{d\chi}| = \frac{a}{2}\sin\left(\frac{\chi}{2}\right), \ \frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\frac{a}{2}\sin\frac{\chi}{2} = \frac{a\cos\frac{\chi}{2}}{2\sin\chi}a\sin\frac{\chi}{2} = \frac{a^2}{4}.$$

The total cross section is

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega}d\Omega = 4\pi^2\frac{a^2}{4} = \pi a^2.$$

Therefore we know the cross section is the effective area of the collisions occur.