Mechanics: Lagrangian Mechanics

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

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I. 2-PARTICLES COUPLED WITH THE SPRINGS

The Lagrangian of the two particles with the same mass m connecting with spring with fix ends is described by the following Lagrangian

$$L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 - \frac{\kappa}{2}(x_1 - l)^2 - \frac{\kappa_{12}}{2}(x_2 - x_1 - l)^2 - \frac{\kappa}{2}(2l - x_2)^2.$$

It is easy to check the equilibrium position is $x_1 = l$, $x_2 = 2l$. Hence we define new coordinates,

$$q_1 = x_1 - l, \;\; q_2 = x_2 - 2l.$$

$$L = \frac{m}{2}\dot{q}_1^2 + \frac{m}{2}\dot{q}_2^2 - \frac{\kappa}{2}q_1^2 - \frac{\kappa_{12}}{2}(q_2 - q_1)^2 - \frac{\kappa}{2}q_2^2.$$

The equations of motion are given as

$$m\ddot{q}_1 = -(\kappa + \kappa_{12})q_1 - \kappa_{12}q_2, \;\; m\ddot{q}_2 = -(\kappa + \kappa_{12})q_2 - \kappa_{12}q_1$$

One can solve those equations by adding and subtracting them,

$$m(\ddot{q}_1 + \ddot{q}) = -(\kappa + 2\kappa_{12})(q_1 + q_2), \;\; m(\ddot{q}_1 - \ddot{q}) = -\kappa(q_1 - q_2).$$

Set $\xi(t)$=$q_1(t) + q_2(t)$ and $\eta(t)$=$q_1(t) - q_2(t)$ we have

$$\begin{aligned}
\xi(t) &= \xi(0)\cos\sqrt{\frac{\kappa + 2\kappa_{12}}{m}}t + \frac{\dot{\xi}(0)}{\sqrt{\frac{\kappa+2\kappa_{12}}{m}}}\sin\sqrt{\frac{\kappa + 2\kappa_{12}}{m}}t, \\
\eta(t) &= \eta(0)\cos\sqrt{\frac{\kappa}{m}}t + \frac{\dot{\eta}(0)}{\sqrt{\frac{\kappa}{m}}}\sin\sqrt{\frac{\kappa}{m}}t,
\end{aligned}$$

$$q_1 = Ae^{i\omega t}, \;\; q_2(t) = Be^{i\omega t}.$$

$$-m\omega^2 A = -(\kappa + \kappa_{12})A - \kappa_{12}B, \;\; -m\omega^2 B = -(\kappa + \kappa_{12})B - \kappa_{12}A$$

$$\begin{pmatrix} -m\omega^2+\kappa+\kappa_{12}, & \kappa_{12} \\ \kappa_{12}, & -m\omega^2+\kappa+\kappa_{12} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determent of this 2×2 matrix must be zero, otherwise we can multiply its inverse matrix and find that $A = B = 0$. Thus we have

$$\det \begin{pmatrix} -m\omega^2+\kappa+\kappa_{12}, & \kappa_{12} \\ \kappa_{12}, & -m\omega^2+\kappa+\kappa_{12} \end{pmatrix} = 0.$$

Therefore we know that

$$-m\omega^2+\kappa+\kappa_{12} = \pm\kappa_{12}, \longrightarrow \omega^2 = \frac{\kappa+2\kappa_{12}}{m},\ \frac{\kappa}{m}.$$

When $\omega^2{=}\frac{\kappa+2\kappa_{12}}{m}$ we have $A{=}B$. On the other hand, when $\omega^2{=}\frac{\kappa}{m}$ we have $A{=}{-B}$. Therefore the solution can be written as the combinations of the following two solutions,

$$\begin{aligned} q_1(t) &= A_+^R \cos\sqrt{\frac{\kappa+2\kappa_{12}}{m}}t - A_+^I \sin\sqrt{\frac{\kappa+2\kappa_{12}}{m}}t,\ \ q_2(t) = q_1(t). \\ q_1(t) &= A_-^R \cos\sqrt{\frac{\kappa}{m}}t - A_+^I \sin\sqrt{\frac{\kappa}{m}}t,\ \ q_2(t) = -q_1(t). \end{aligned}$$

II. 3-PARTICLES COUPLED WITH THE SPRINGS

Now if we increase number of particles then the Lagrangian becomes

$$L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 - \frac{\kappa}{2}(x_1-l)^2 - \frac{\kappa}{2}(x_2-x_1-l)^2 - \frac{\kappa}{2}(x_3-x_2-l)^2 - \frac{\kappa}{2}(3l-x_3)^2.$$

We change to new coordinates,

$$q_1 = x_1 - l,\ q_2 = x_2 - 2l,\ \ q_3 = x_3 - 3l.$$

The Lagrangian becomes

$$L = \frac{m}{2}\dot{q}_1^2 + \frac{m}{2}\dot{q}_2^2 + \frac{m}{2}\dot{q}_3^2 - \frac{\kappa}{2}q_1^2 - \frac{\kappa}{2}(q_2-q_1)^2 - \frac{\kappa}{2}(q_3-q_2)^2 - \frac{\kappa}{2}q_3^2.$$

The equations of the motion become

$$\begin{aligned} m\ddot{q}_1 &= -2\kappa q_1 + \kappa q_2, \\ m\ddot{q}_2 &= \kappa q_1 - 2\kappa q_2 + \kappa q_3, \\ m\ddot{q}_1 &= -2\kappa q_1 + \kappa q_2, \end{aligned}$$

We look for normal modes, so we set

$$q_1(t) = Ae^{i\omega t},\ q_2(t) = Be^{i\omega t},\ q_3(t) = Ce^{i\omega t}.$$

Thus one has equations like these ones.

$$\begin{pmatrix} -m\omega^2 + 2\kappa, & -\kappa, & 0 \\ -\kappa, & -m\omega^2 + 2\kappa, & -\kappa \\ 0 & -\kappa & -m\omega^2 + 2\kappa \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The characteristic frequencies must satisfy the following equation:

$$\det \begin{pmatrix} -m\omega^2 + 2\kappa, & -\kappa, & 0 \\ -\kappa, & -m\omega^2 + 2\kappa, & -\kappa \\ 0 & -\kappa & -m\omega^2 + 2\kappa \end{pmatrix} = 0$$

Set $-m\omega^2 + 2\kappa$=Δ, the equation becomes

$$\Delta^3 - 2\kappa^2\Delta = 0.$$

Therefore we know that

$$\Delta = 0, \sqrt{2}\kappa, -\sqrt{2}\kappa.$$

$$\omega^2 = \frac{2\kappa}{m}, \frac{(2+\sqrt{2})\kappa}{m}, \frac{(2-\sqrt{2})\kappa}{m}.$$

When ω^2=$\frac{2\kappa}{m}$, we have

$$\begin{pmatrix} 0 & -\kappa, & 0 \\ -\kappa, & 0 & -\kappa \\ 0 & -\kappa & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is easy to see the relation between A B and C is A=-C, B=0. When ω^2=$\frac{(2\pm\sqrt{2})\kappa}{m}$, we have

$$\begin{pmatrix} \pm\sqrt{2}\kappa & -\kappa, & 0 \\ -\kappa, & \pm\sqrt{2}\kappa & -\kappa \\ 0 & -\kappa & \pm\sqrt{2}\kappa \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is easy to see the relations between A B and C are $A{=}C{=}\pm\frac{B}{\sqrt{2}}$. Therefore there exist three independent solutions:

$$
\begin{aligned}
q_1(t) &= A_0^R \cos\sqrt{\frac{2\kappa}{m}}t - A_0^I \sin\sqrt{\frac{2\kappa}{m}}t,\ \ q_2(t) = 0,\ \ q_3(t) = -q_1(t). \\
q_1(t) &= A_+^R \cos\sqrt{\frac{(2+\sqrt{2})\kappa}{m}}t - A_+^I \sin\sqrt{\frac{(2+\sqrt{2})\kappa}{m}}t,\ \ q_2(t) = \sqrt{2}q_1(t),\ \ q_3(t) = q_1(t). \\
q_1(t) &= A_-^R \cos\sqrt{\frac{(2-\sqrt{2})\kappa}{m}}t - A_+^I \sin\sqrt{\frac{(2-\sqrt{2})\kappa}{m}}t,\ \ q_2(t) = -\sqrt{2}q_1(t),\ \ q_3(t) = q_1(t).
\end{aligned}
$$

The general solutions are the linear combination of these three ones.