Mechanics: Lagrangian Mechanics

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I. N-COUPLED PARTICLES

The Lagrangian of N coupled particle is

$$L = \sum_{i=0}^{N} \frac{m}{2}\dot{x}_i^2 - \frac{\kappa}{2}(x_{i+1} - x_i - l)^2.$$

Here we set x_0=0 and x_{N+1}=$(N+1)l$. The equilibrium point is x_k=kl. Here k=0,1,2,...N+1. Therefore we set q_k=$x_k - kl$. The Lagrangian becomes

$$L = \sum_{i=0}^{N} \frac{m}{2}\dot{q}_i^2 - \frac{\kappa}{2}(q_{i+1} - q_i)^2$$

Of course q_0=q_{N+1}=0. The equations of motion are

$$m\ddot{q}_i = \kappa q_{i-1} - 2\kappa q_i + \kappa q_{i+1}.$$

i runs from 1 to N. To obtain the normal mode, we assume that q_i=$A_i e^{i\omega t}$. Then we have the following equations,

$$\kappa A_{i-1} + \tau A_i + \kappa A_{i+1} = 0.$$

Here τ=$m\omega^2 - 2\kappa$. One may think that it should first obtain the characteristic frequency then obtain the relations between A_i. However here we provide an alternative way to solve this problem. The trick is to rewrite the equations as follows,

$$\kappa(A_{i-1} + \alpha A_i) = \beta(A_i + \alpha A_{i+1}).$$

It is straightforward to get those relations,

$$\alpha\kappa - \beta = \tau, \ \ -\alpha\beta = \kappa.$$

Then the particular combination of the two terms can be written as follows,

$$B_i = A_i + \alpha A_{i+1} = \left(\frac{\kappa}{\beta}\right)^i \alpha A_1 = (-\alpha)^i \alpha A_1.$$

By simply algebra one can obtain A_i from B_i's,

$$\sum_{k=0}^{k=i-1} (-\alpha)^k B_k = (-\alpha)^{i-1} \alpha A_i.$$

Insert the expression of B_i one has

$$\sum_{k=0}^{k=i-1} (-\alpha)^k B_k = \sum_{k=0}^{k=i-1} (-\alpha)^k (-\alpha)^k \alpha A_1 = \sum_{k=0}^{k=i-1} \alpha^{2k} \alpha A_1 = A_1 \left(\frac{\alpha^{2i+1} - \alpha}{\alpha^2 - 1} \right).$$

Hence we have the following result:

$$A_i = \frac{1}{\alpha(-\alpha)^{i-1}} \left(\frac{\alpha^{2i+1} - \alpha}{\alpha^2 - 1} \right) A_1 = (-1)^{i-1} \left(\frac{\alpha^{i+1} - \alpha^{-i+1}}{\alpha^2 - 1} \right) A_1.$$

Assume $\tau < 0$. Set $\alpha{=}-\cos\gamma - i\sin\gamma{=}-e^{i\gamma}$, $\beta{=}\kappa e^{-i\gamma}$. Then $\tau{=}-2\kappa\cos\gamma$. Therefore

$$\begin{aligned} A_j &= (-1)^{j-1} \left(\frac{(-1)^{j+1} e^{i(j+1)\gamma} - (-1)^{-j+1} e^{-i(j-1)\gamma}}{e^{2i\gamma} - 1} \right) A_1 = \left(\frac{e^{ij\gamma} - e^{-ij\gamma}}{e^{i\gamma} - e^{-i\gamma}} \right) A_1. \\ &= \left(\frac{\sin j\gamma}{\sin\gamma} \right) A_1. \end{aligned}$$

$A_{N+1}{=}0$. Therefore $\sin(N+1)\gamma{=}0$. It is easy to find that $\gamma{=}\frac{k\pi}{N+1}$. Here k=1,2...N. The remain step is to determine the values of ω^2. Remember that

$$m\omega^2 - 2\kappa = \tau = -2\kappa\cos\gamma. \longrightarrow \omega^2 = \frac{2\kappa(1-\cos\gamma)}{m} = \frac{4\kappa}{m}\sin^2\frac{\gamma}{2}. \Longrightarrow \omega = \pm 2\sqrt{\frac{\kappa}{m}}\sin\frac{\gamma}{2}.$$

Therefore we reach the conclusion that when the k-th normal mode is the mode corresponding to $\omega_k{=}\pm 2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{k\pi}{2(N+1)}\right)$. If we choose its A_1 as $\sin\left(\frac{k\pi}{N+1}\right)$ then we have

$$A_j = \sin\left(\frac{jk\pi}{N+1}\right). \quad j = 1, 2.....N$$