Mechanics: Lagrangian Mechanics

Chung Wen Kao
Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

(Dated: January 1, 2013)

This lecture was given on Dec 21st.

I. EQUATION OF MOTION IN THE NON-INERTIAL FRAMES

$$\left(\frac{d\vec{A}}{dt}\right)_{fix} = \left(\frac{d\vec{A}}{dt}\right)_{rot} + \vec{\omega}\times\vec{A}.$$

$$\frac{d\hat{e}}{dt} = \vec{\omega}\times\hat{e}.$$

$\vec{\omega}=\omega_3\hat{e}_3$.

$$\hat{e}_1 = \cos(\omega_3 t)\hat{e}_{1,f} + \sin(\omega_3 t)\hat{e}_{2,f}, \;\; \hat{e}_2 = -\sin(\omega_3 t)\hat{e}_{1,f} + \cos(\omega_3 t)\hat{e}_{2,f}.$$

$$\frac{d\hat{e}_1}{dt} = \omega_3(-\sin(\omega_3 t)\hat{e}_{1,f}+\cos(\omega_3 t)\hat{e}_{2,f}) = \omega_3\hat{e}_2, \;\; \frac{d\hat{e}_2}{dt} = \omega_3(-\cos(\omega_3 t)\hat{e}_{1,f}-\sin(\omega_3 t)\hat{e}_{2,f}) = -\omega_3\hat{e}_1.$$

$$\frac{d\hat{e}_i}{dt} = \epsilon_{ijk}\omega_j\hat{e}_k \Longrightarrow \frac{d\hat{e}}{dt} = \vec{\omega}\times\hat{e}.$$

$$\begin{aligned}\vec{A} &= A_i\hat{e}_i, \;\; \frac{d\vec{A}}{dt} = \frac{dA_i}{dt}\hat{e}_i + A_i\frac{d\hat{e}_i}{dt} = \frac{dA_i}{dt}\hat{e}_i + A_i\epsilon_{ijk}\omega_j\hat{e}_k \\ &= \frac{dA_i}{dt}\hat{e}_i + \hat{e}_k\epsilon_{kij}A_i\omega_j = \left(\frac{d\vec{A}}{dt}\right)_{rot} + \vec{\omega}\times\vec{A}.\end{aligned}$$

.

$$\begin{aligned}\frac{1}{m}\vec{F} &= \left(\frac{d^2\vec{r}}{dt^2}\right)_{fix} = \frac{d}{dt}\left(\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}\right)_{fix} \\ &= \left(\frac{d\vec{v}_{rot}}{dt}\right)_{fix} + \frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)_{fix} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + \vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{fix} \\ &= \vec{a}_{rot} + \vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{v}_{rot}+\vec{\omega}\times\vec{r}) = \vec{a}_{rot} + 2\vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r}).\end{aligned}$$

II. EULER EQUATION OF RIGID BODY

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{fix} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega} \times \vec{L}.$$

$$\vec{\tau} = \sum_{i=1}^{3} \tau_i \hat{e}_i,\;\; \vec{\omega} = \sum_{i=1}^{3} \omega_i \hat{e}_i,\;\; \vec{L} = \sum_{i=1}^{3} I_i \omega_i \hat{e}_i,\;\; \left(\frac{d\vec{L}}{dt}\right)_{rot} = \sum_{i=1}^{3} I_i \dot{\omega}_i \hat{e}_i.$$

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= \tau_1, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= \tau_2, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= \tau_3, \end{aligned}$$

III. MOTION OF SYMMETRICAL TOP WITHOUT EXTERNAL TORQUE

If $\vec{\tau}$=0. I_1=I_2.

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3 \Longrightarrow \omega_3 = constant.$$

Hence the equations become,

$$\dot{\omega}_1 + \Omega\omega_2 = 0,\;\; \dot{\omega}_2 - \Omega\omega_1 = 0.$$

Here $\Omega = \frac{I_3 - I_1}{I_1}\omega_3$. Set $\eta = \omega_1 + i\omega_2$.

$$\dot{\eta} - i\Omega\eta = 0.$$

$\eta(t) = \eta(0)e^{i\Omega t}$. $\Longrightarrow \omega_1(t) + i\omega_2(t) = [\omega_1(0)\cos(\Omega t) - \omega_2(0)\sin(\Omega t)] + i[\omega_2(0)\cos(\Omega t) + \omega_1(0)\sin(\Omega t)]$.

$$\omega_1(t) = \omega_1(0)\cos(\Omega t) - \omega_2(0)\sin(\Omega t),\;\; \omega_2(t) = \omega_2(0)\cos(\Omega t) + \omega_1(0)\sin(\Omega t).$$

$$|\eta(t)|^2 = \eta(t)\eta^*(t) = |\eta(0)|^2, \Longrightarrow \omega_1^2(t) + \omega_2^2(t) = constant.$$

Since ω_3 is a constant, hence $|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$ is also a constant.