Mechanics: Lagrangian Mechanics

Chung Wen Kao

Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

(Dated: December 29, 2012)

This lecture was given on Dec 28th.

I. MOTION OF SYMMETRIC TOP WITH ONE POINT FIXED

$$T = \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2 = \frac{1}{2}I_1(\omega_1^2+\omega_2^2) + \frac{1}{2}I_3\omega_3^2.$$

$$\omega_1^2+\omega_2^2 = (\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)^2 + (\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)^2 = \dot\phi\sin^2\theta + \dot\theta^2.$$

$$L = \frac{I_1}{2}(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{I_3}{2}(\dot\phi\cos\theta + \dot\psi)^2 - Mgh\cos\theta.$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta = constant.$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3 = constant.$$

$$E = \frac{I_1}{2}(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{I_3}{2}(\dot\phi\cos\theta + \dot\psi)^2 + Mgh\cos\theta.$$

$$\dot\psi = \frac{p_\psi - I_3\dot\phi\cos\theta}{I_3}.$$

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}.$$

$$\dot\psi = \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}.$$

$$
\begin{aligned}
E &= \frac{I_1}{2}(\dot\phi^2\sin^2\theta+\dot\theta^2)+\frac{I_3}{2}(\dot\phi\cos\theta+\dot\psi)^2+Mgh\cos\theta \\
&= \frac{I_1}{2}(\dot\phi^2\sin^2\theta+\dot\theta^2)+\frac{p_\psi^2}{I_3}+Mgh\cos\theta,
\end{aligned}
$$

$$E' = E-\frac{p_\psi^2}{I_3}=\frac{I_1}{2}\dot\theta^2+\frac{(p_\phi-p_\psi\cos\theta)^2}{2I_1\sin^2\theta}+Mgh\cos\theta.$$

$$V(\theta)=\frac{(p_\phi-p_\psi\cos\theta)^2}{2I_1\sin^2\theta}+Mgh\cos\theta.$$

x=$\cos\theta$.

$$V(x)=\frac{(p_\phi-p_\psi x)^2}{2I_1(1-x^2)}+Mghx.$$

$$t(\theta)=\int\frac{d\theta}{\sqrt{\frac{2}{I_1}\left[E'-V(\theta)\right]}}.$$

$$\frac{dV}{d\theta}=\frac{-\cos\theta(p_\phi-p_\psi\cos\theta)^2+p_\psi\sin^2\theta(p_\phi-p_\psi\cos\theta)}{I_1\sin^3\theta}-Mgh\sin\theta.$$

β=$p_\phi-p_\psi\cos\theta_0$.

$$\cos\theta_0\beta^2-p_\psi\sin^2\theta_0\beta+MghI_1\sin^4\theta_0=0.$$

$$\beta=\frac{p_\psi\sin^2\theta_0}{2\cos\theta_0}\left(1\pm\sqrt{\frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right).$$

$$1\pm\sqrt{\frac{4MghI_1\cos\theta_0}{p_\psi^2}}\geq 0.\Longrightarrow p_\phi^2\leq 4MghI_1\cos\theta_0\Longrightarrow\omega_3\geq\frac{2}{I_3}\sqrt{MghI_1\cos\theta_0}.$$

$$\dot\phi_0=\frac{\beta}{I_1\sin^2\theta_0}.$$

If $\frac{4MghI_1\cos\theta_0}{p_\psi^2}\ll 1$ then

$$
\begin{aligned}
\beta_+ &\sim \frac{p_\psi\sin^2\theta_0}{\cos\theta_0}\longrightarrow\dot\phi_0^+\sim\frac{I_3\omega_3}{I_1\cos\theta_0}, \\
\beta_- &\sim \frac{p_\psi\sin^2\theta_0}{\cos\theta_0}\frac{2MghI_1\cos\theta_0}{p_\psi^2}=\frac{MghI_1\sin^2\theta_0}{I_3\omega_3}\longrightarrow\dot\phi_0^-\sim\frac{Mgh}{I_3\omega_3}.
\end{aligned}
$$