Beginning Supersymmetry (SUSY)

An Elementary Introduction to Supersymmetric Quantum Mechanics

Ref: ITEP Lectures on Particle Physics and Field Theory Vol. 1 by M. Shifman

Outline

1. Introduction
2. A "Physical" Realization of SUSY
3. Construction of SUSY Invariant Actions Based on Superfield Language
4. Spontaneous Breakdown of SUSY
5. Summary

What is SUSY = supersymmetry? $[Q, H] = 0$

1. Putting bosons & fermions on equal footing.
   (degeneracy ⇒ multiplet) $Q|F\rangle \sim |B\rangle$, $Q|B\rangle \sim |F\rangle$
2. Nontrivial unification of space-time & internal symmetries
   (No-Go theorem ⇒ graded Lie algebra)
3. A solution for improvement of ultraviolet behaviour of quantum field theory
   (gauge hierarchy, cosmological constant)
4. A useful technique for attacking non-perturbative dynamics of gauge theories.

A "Physical" Realization of Supersymmetry

1-dim electron moving in a magnetic field background

$$\hat{H} = \frac{1}{2}(\vec{p} - \vec{A})^2 + \frac{1}{2}\vec{\sigma}\cdot\vec{B} \,, \quad \vec{B} = \vec{\nabla}\times\vec{A} = W'(x)\hat{e}_z$$

choose vector potential $A_x = A_z = 0$. $A_y = A_y(x) = W(x)$

$$\hat{H} = \frac{1}{2}(\vec{p}^{\,2} - \vec{p}\cdot\vec{A} - \vec{A}\cdot\vec{p} + \vec{A}^2) + \frac{1}{2}\sigma_z W'$$

one-dimensional electron $\Rightarrow$ $p_y = p_z = 0$

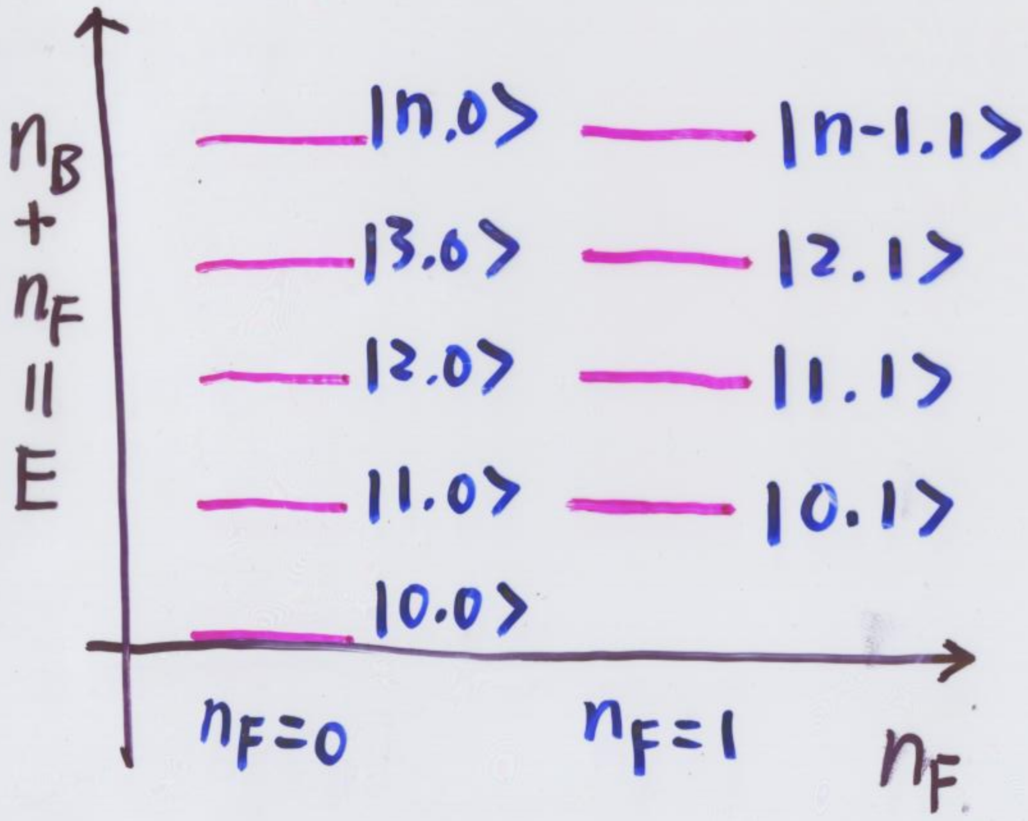
$$\hat{H} = \frac{1}{2}p_x^2 + \frac{1}{2}W^2(x) + \frac{1}{2}\sigma_z W'(x)$$

special case: $W(x) = x$ (constant magnetic field)

Spectrum of $\hat{H} = \frac{1}{2}(p^2 + x^2) + \frac{1}{2}\sigma_z$

$$|n_B, n_F\rangle \equiv \frac{1}{\sqrt{n_B!}}(a^+)^{n_B}|0, n_F\rangle \quad \begin{cases} n_B = 0, 1, 2, \ldots \\ n_F = 0, 1 \end{cases}$$

$|0, 0\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (spindown), $|0, 1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin up)



$$\hat{H}|n_B, n_F\rangle = (n_B + n_F)|n_B, n_F\rangle$$

$$\frac{1}{2}\sigma_3|n_B, n_F\rangle = (n_F - \tfrac{1}{2})|n_B, n_F\rangle$$

$$n_F = \frac{1}{2}(1 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{Q} \equiv \frac{1}{2}(\sigma_1 \hat{p} + \sigma_2 \hat{x})$$

$$\hat{Q}|n_B, 0\rangle \sim |n_B - 1, 1\rangle$$

$$\tilde{Q}|n_B, 1\rangle \sim |n_B + 1, 0\rangle$$

General Case: Define the Supercharge

$$Q_1 \equiv \frac{1}{2}(\sigma_1 \hat{p} + \sigma_2 W) = Q_1^\dagger$$

$$Q_2 \equiv \frac{1}{2}(\sigma_2 \hat{p} - \sigma_1 W) = Q_2^\dagger = -i\sigma_3 Q_1$$

one have the following graded algebra

$$\{Q_i, Q_j\} = \delta_{ij} H, \quad \underbrace{[Q_i, H] = 0}_{\text{symmetry charge}} \quad . \; i, j = 1$$

$$\begin{cases} Q_i : \text{odd element}, \; i = 1, 2 \\ H : \text{even element} \end{cases}$$

$\Rightarrow$ SUSY generators = square roots of usual (bosonic) space-time translations.

Essential Features of SUSY

① conserved quantity = $\frac{d}{dt}Q_k = -i[Q_k, \hat{H}] = 0$

② degeneracy of excited spectrum

if $\hat{H}|\psi\rangle = E|\psi\rangle \Rightarrow |\varphi_i\rangle \equiv Q_i|\psi\rangle,\ \hat{H}|\varphi_i\rangle = E|\varphi_i\rangle$

③ semi-positive-definiteness of energy spectrum

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|Q^{\dagger}Q|\psi\rangle = \|Q|\psi\rangle\|^2 \geq 0$$

④ vanishing ground-state energy

if $\langle\Omega|\hat{H}|\Omega\rangle = 0 \Leftrightarrow \hat{Q}_i|\Omega\rangle = \hat{H}|\Omega\rangle = 0$

$\Rightarrow \{$ S.S.B. of SUSY $\Leftrightarrow$ existence of NORMALIZABLE zero-energy state

⑤ NO perturbative correction to classical ground state energy

Example: $W(x) = x(1-\lambda x^2)$

$$\Rightarrow \hat{H} = \frac{1}{2}\left[p^2 + x^2(1-\lambda x^2)^2\right] + \frac{\sqrt{3}}{2}(1-3\lambda x^2)$$

$$= \frac{1}{2}(p^2 + x^2 + \sqrt{3}) - \lambda\left(x^4 + \frac{3}{2}\sqrt{3}x^2\right) + O(\lambda^2)$$

(i) unperturbed ground-state

$$|0,0\rangle = \begin{pmatrix} 0 \\ \phi_0(x) \end{pmatrix}, \quad \phi_0(x) = \sqrt[4]{\frac{1}{\pi}}\, e^{-\frac{1}{2}x^2}$$

(ii) unperturbed ground-state energy

$$\langle 0,0|H_0|0,0\rangle = \frac{1}{2} - \frac{1}{2} = 0$$

$f(x) = e^{-\frac{1}{x^2}}$

(iii) 1st-order correction

$$\Delta E = \lambda \int_{-\infty}^{\infty} dx\, \phi_0^2(x)\left\{-x^4 + \frac{3}{2}x^2\right\} = 0$$

persist to all order in λ!!!

Quick Review of Grassmann Algebra & Calculus

① anticommuting product $\theta_i \theta_j = -\theta_j \theta_i$, $\theta_i^2 = 0$.

② Taylor Expansion , $f(\theta) = a + \zeta\theta$

$$g(\theta_1, \theta_2) = g_0 + \psi_1\theta_1 + \psi_2\theta_2 + a\theta_1\theta_2$$

$\Rightarrow$ "analytic function" in Grassmann variables

$\Rightarrow$ FINITE polynomial

③ differentiation $\frac{d}{d\theta_i} 1 = 0$. $\frac{d}{d\theta_i}\theta_j = \delta_{ij}$

$$\frac{d}{d\theta_i}(\theta_j \theta_k) = \delta_{ij}\theta_k - \delta_{ik}\theta_j$$

④ integration $\int d\theta_i = 0$, $\int d\theta_i \theta_j = \delta_{ij}$

⑤ complex notation $\theta = \theta_1 + i\theta_2$. $\bar{\theta} \equiv i(\theta_1 - i\theta_2)$

$\Rightarrow \zeta\bar{\theta} = (\zeta \times \theta) + i(\zeta\,\theta)$

$\theta\zeta = (\theta \times \zeta) + i(\theta\,\zeta) = (\zeta \times \theta) - i(\zeta\theta)$

"Space-Time" Realization of Supersymmetry

Introduce the notion of superspace (or supertime)

$$t \to (t, \theta, \bar{\theta}),$$

the supertime-translation is defined as

$$\theta \to \theta + \xi, \quad \bar{\theta} \to \bar{\theta} + \bar{\xi}, \quad t \to t + i(\theta\bar{\xi} - \xi\bar{\theta}).$$

One can check that these transformations form a group by checking the composition rule.

$$\theta \to \theta + \xi_1 + \xi_2, \quad \bar{\theta} \to \bar{\theta} + \bar{\xi}_1 + \bar{\xi}_2$$

$$t \to t + i[\theta(\bar{\xi}_1 + \bar{\xi}_2) - (\xi_1 + \xi_2)\bar{\theta}] + i(\xi_2\bar{\xi}_1 - \xi_1\bar{\xi}_2)$$

A linear realization of SUSY algebra!

Having defined the "super-time transformation"

$$\theta \to \theta + \xi, \quad \bar{\theta} \to \bar{\theta} + \bar{\xi}, \quad t \to t + i(\theta\bar{\xi} - \xi\bar{\theta})$$

We can derive (define) the covariant derivatives in analogus to the usual Taylor Expansion

$$f(t+\Delta t) = f(t) + (\Delta t) f'(t) + \frac{(\Delta t)^2}{2} f''(t) + \cdots$$

$$= e^{(\Delta t)\frac{d}{dt}} f(t)$$

$$\Rightarrow e^{\xi \frac{\partial}{\partial \theta} + \bar{\xi} \frac{\partial}{\partial \bar{\theta}} + i(\xi\bar{\theta} - \theta\bar{\xi})\frac{\partial}{\partial t}} = e^{\xi \mathcal{D} + \bar{\mathcal{D}} \bar{\xi}}$$

where $\mathcal{D} \equiv \frac{\partial}{\partial \theta} + i\bar{\theta}\frac{\partial}{\partial t}$ $\qquad \{\mathcal{D}, \mathcal{D}\} = \{\bar{\mathcal{D}}, \bar{\mathcal{D}}\} = 0$

$\bar{\mathcal{D}} \equiv -\frac{\partial}{\partial \bar{\theta}} - i\theta\frac{\partial}{\partial t}$ $\qquad \{\mathcal{D}, \bar{\mathcal{D}}\} = -2i\frac{\partial}{\partial t}$

Complete SUSY Transformations of Superfield

$$\Phi(t', \theta', \bar{\theta}') = \tilde{\phi}_\xi(t) + \theta\,\tilde{\bar{\psi}}_\xi(t) + \tilde{\psi}_\xi(t)\,\bar{\theta} + \tilde{D}_\xi(t)\,\theta\bar{\theta}$$

$$\delta\phi(t) \equiv \tilde{\phi}_\xi(t) - \phi(t) = \xi\bar{\psi}(t) + \psi(t)\bar{\xi} + D(t)\,\xi\bar{\xi}$$

$$\delta\psi(t) \equiv \tilde{\psi}_\xi(t) - \psi(t) = \xi\left[-i\left(\frac{d\phi}{dt}\right) + D(t)\right] + i\left(\frac{d\psi}{dt}\right)\xi\bar{\xi}$$

$$\delta\bar{\psi}(t) \equiv \bar{\tilde{\psi}}_\xi(t) - \bar{\psi}(t) = \bar{\xi}\left[i\left(\frac{d\phi}{dt}\right) + D(t)\right] - i\left(\frac{d\bar{\psi}}{dt}\right)\xi\bar{\xi}$$

$$\delta D(t) \equiv \tilde{D}_\xi(t) - D(t) = i\left[\xi\frac{d\bar{\psi}}{dt} - \frac{d\psi}{dt}\bar{\xi}\right] - \frac{d^2\phi}{dt^2}\xi\bar{\xi}$$

Notice that $\delta D(t)$ is a total derivative (in time)!

How to Construct a SUSY-invariant Action?

$$\delta S_{SUSY} = \int dt\, \delta L_{SUSY}(\phi, \dot{\phi}; \psi, \bar{\psi})$$

$$\{\delta S_{SUSY} = 0\} \Longleftrightarrow \{\delta L_{SUSY} = \text{total derivative int}\}$$

Hints: (1) $\delta D(t) = \frac{d}{dt}\{\xi\bar{\psi} - \psi\bar{\xi} - \phi\xi\bar{\xi}\}$

(2) $\int d\bar{\theta}\, d\theta\, \Phi(t) = D(t)$

(3) Algebra + Calculus of superfields

$$\Rightarrow S_{SUSY} = \int dt\, d\bar{\theta}\, d\theta \left[\frac{1}{2}(\bar{\mathcal{D}}\Phi)(\mathcal{D}\Phi) - F(\Phi)\right]$$

$F(\Phi)$: Superpotential.

One can check that

① $\frac{1}{2}\int dt\, d\bar{\theta}\, d\theta\, (\bar{\mathcal{D}}\Phi)_{tch} (\mathcal{D}\Phi)_{tach}$

$= \int dt \left[\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}D^2 + i\frac{d\psi}{dt}\bar{\psi} \right]$

② $\int dt\, d\bar{\theta}\, d\theta\, F(\Phi) = \int dt \left[F'(\phi) D + F''(\phi)\bar{\psi}\psi \right]$

$\Rightarrow L_{susy} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}D^2 - i\bar{\psi}\frac{d\psi}{dt} - F'D - F''\bar{\psi}\psi$

③ D is an auxilary field, can be integrated out!

④ $\{\psi, \psi\} = \{\bar{\psi}, \bar{\psi}\} = 0$, $\{\psi, \bar{\psi}\} = 1$. choose $\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow H_{susy} = \frac{p^2}{2} + \frac{1}{2}(F')^2 + \frac{1}{2}\sigma_3 (F'') \Rightarrow \boxed{F' = W}$

Spontaneous Breakdown of Supersymmetry

A Necessary and Sufficient condition of SUSY invariance $\Leftrightarrow$ Existence of Normalizable g.s.

Solution to the 1st order differential equation $Q|\Omega\rangle = 0$

$$\left[\sigma_1 \hat{p} + \sigma_2 W(x)\right]|\Omega\rangle = 0, \quad \langle x, \vec{\sigma}|\Omega\rangle = \exp\left[\int_0^x W(y)\,dy\,\sigma_3\right]\begin{pmatrix} u \\ v \end{pmatrix}$$

where $\langle 0, \vec{\sigma}|\Omega\rangle = \begin{pmatrix} u \\ v \end{pmatrix}$, $\langle 0, \downarrow|\Omega\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$, $\langle 0, \uparrow|\Omega\rangle = \begin{pmatrix} u \\ 0 \end{pmatrix}$

Normalizable State $\Rightarrow$ $W(-\infty)\,W(\infty) < 0$

Example : $W(y) = y$, choose $\langle 0, \vec{\sigma}|\Omega\rangle = \frac{1}{\pi^{1/4}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\langle \Omega|\Omega\rangle = \int dx \left(\frac{1}{\pi^{1/2}}\right) e^{-x^2} = 1 .$$

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The condition $W(-\infty)\,W(\infty) < 0$ implies that

$W(x)$ has odd # of zeros = sum of $\begin{cases} \text{zero of } W \text{ with } W' > 0 \\ \text{zero of } W \text{ with } W' < 0 \end{cases}$

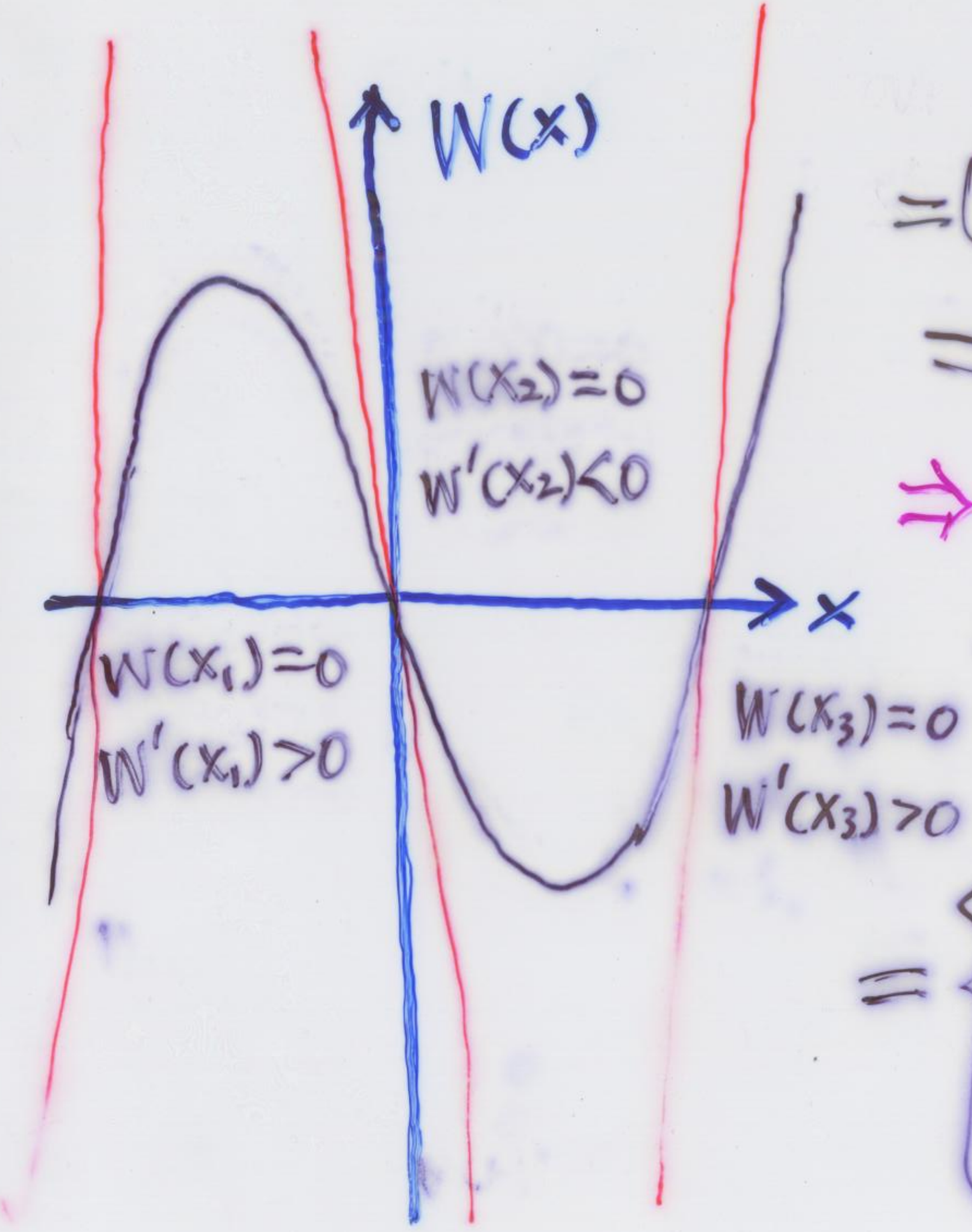
and this property is invariant under continuous deformation of $W(x)$ (varying the parameters in W)

$\Rightarrow$ In particular, we can deform the shape of $W(x)$

near zero of W with $W' > 0$ $\Rightarrow$ we have a local bosonic g.s

near zero of W with $W' < 0$ $\Rightarrow$ we have a local fermionic g.s.

$\Rightarrow$ enhancement of the strength of $W(x)$ between zeros. forbid tunneling effects among local g.s



Sum of zeros of $W(x)$

$= (\#\text{ of bosonic g.s}) + (\#\text{ of fermonic g.s})$

$= \text{ODD}$

$\Rightarrow I_W \equiv n_B^{E=0} - n_F^{E=0}$ (Witten index)

$= (\#\text{ of } W(x)=0, \text{ with } W'>0)$
$- (\#\text{ of } W(x)=0, \text{ with } W'<0)$

$$= \begin{cases} 1 & W(\infty)>0,\ W(-\infty)<0 \\ 0 & W(\infty)\cdot W(-\infty)>0 \\ -1 & W(\infty)<0,\ W(-\infty)>0 \end{cases}$$

So Witten Index provides a NECESSARY condition for deciding whether SUSY is spontaneously broken.

A non-zero Witten index $\Rightarrow$ Exact SUSY

Pro & Con of SSB for SUSY.

1. ☹ needs to rely on non-perturbative mechanism (due to non-renormalization theorem)

2. ☺ SUSY can be spontaneously broken even with finite d.o.f. (contrast. global sym in field theory).