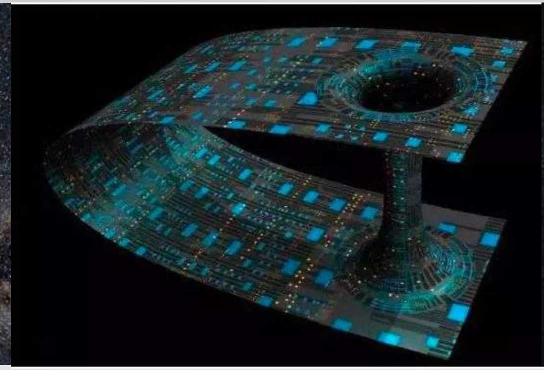
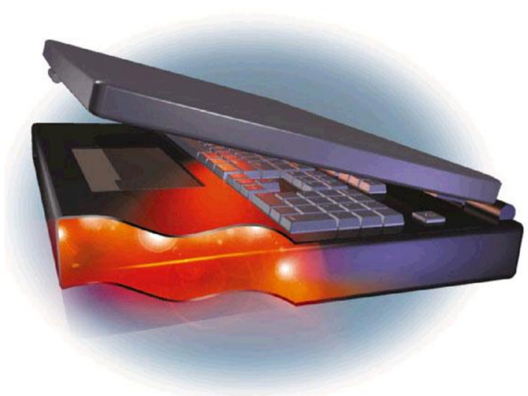
  

Complexity in holographic duality and quantum field theories



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Outline

- What the “complexity” is and why we study it;
- Complexity in AdS/CFT ;
- Attempts to define complexity in QFTs;
- Current challenges

- Knowing the basic principles does not meaning we can understand the world!

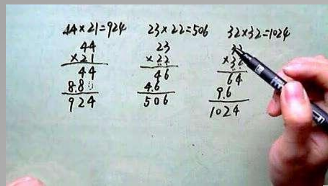
quantum mechanics | spacetime | gravity

$$W = \int_{k<\Lambda} [Dg][DA][D\psi][D\Phi]\, \exp\left\{ i\int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\psi}^i\gamma^\mu D_\mu\psi^i + \left(\bar{\psi}^i_L V_{ij}\Phi\psi^j_R + \mathrm{h.c.}\right) - |D_\mu\Phi|^2 - V(\Phi)\right]\right\}$$

other forces | matter | Higgs

Computing

Computing connects theories and realities



One basic question:

Does the computing speed have upper limit?

Computing is faster and faster?

Complexity and computation speed

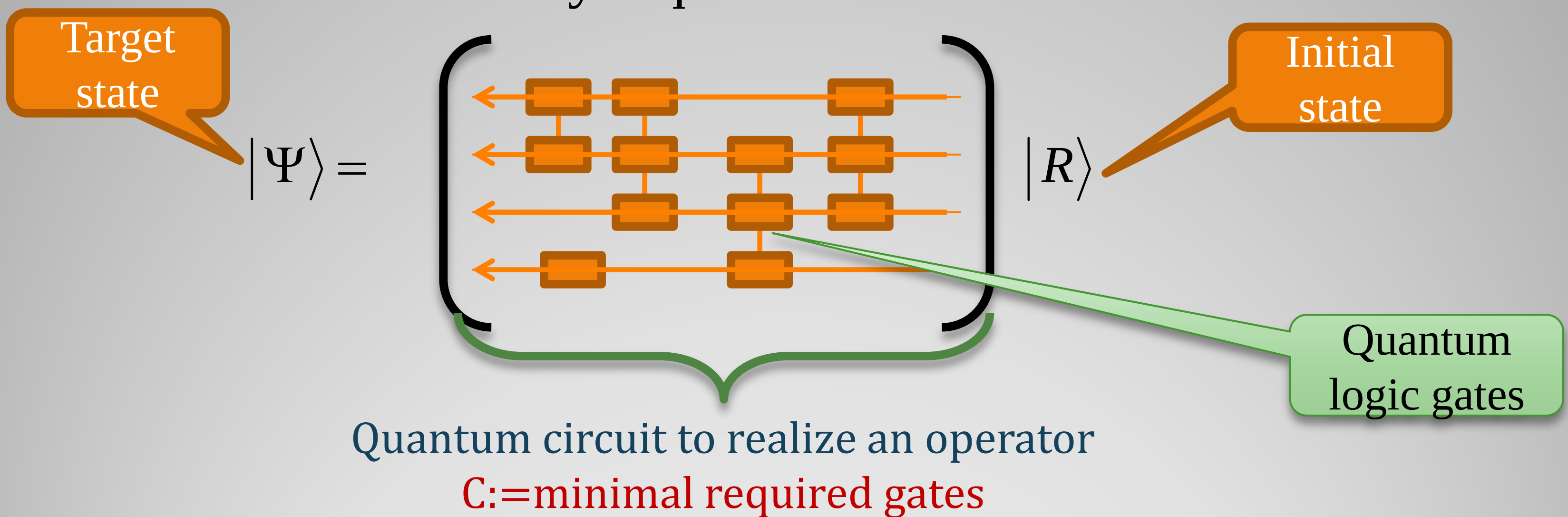
- Let ΔC be the amount of computation task, T is the time to finish the this task, then computation speed is

$$v = \frac{\Delta C}{T}$$

- What is the meaning of “amount of computation” ?

Complexity is a quantity to measure the “amount of computation”

- In quantum circuits, the complexity is usually defined in the finite (discreet) Hilbert space.
- Two states can be associated by an unitary operator $\hat{U}$ which could be simulated by a quantum circuit.



- Complexity for one operator is defined as the minimal required gates when we realized this operator by quantum circuits.

Complexity in quantum circuits

- To understand the circuit complexity, let's consider a quantum circuit.

- A quantum circuit (QC) is a device composed of qubits and gates (called g) whose purpose is to implement special unitary transformations on an initial state of the qubits.

$$\left|0_i\right\rangle = \left|\uparrow_i\right\rangle \xrightarrow{g_i} \left|1_i\right\rangle = \left|\downarrow_i\right\rangle$$

- For example, considering a K-spin system:

$$\left|101011\cdots1\right\rangle = g_1 g_3 g_4 g_5 \cdots g_K \left|0000\cdots0\right\rangle = O(n)\left|0000\cdots0\right\rangle$$

- The quantum gate complexity is **minimal** gates required to obtain target state from reference state.

- What is the fundamental gates?
- In computer science, we can use the universal gates set, such {NOT,OR,AND} and so on.
- But this makes the results lose the exact physical sense.
- It cannot be used in continuous systems.

The problems in this native ideas

holographic duality

- Holographic duality(AdS/CFT):

CFT in d dim = gravity in (d+1) dim

It’s complicated in algebras

Geometrization of a quantum CFT state

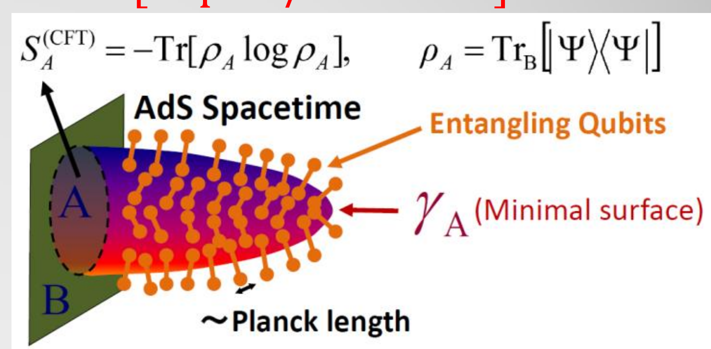
- A powerful tool to study non-gravity system such as:

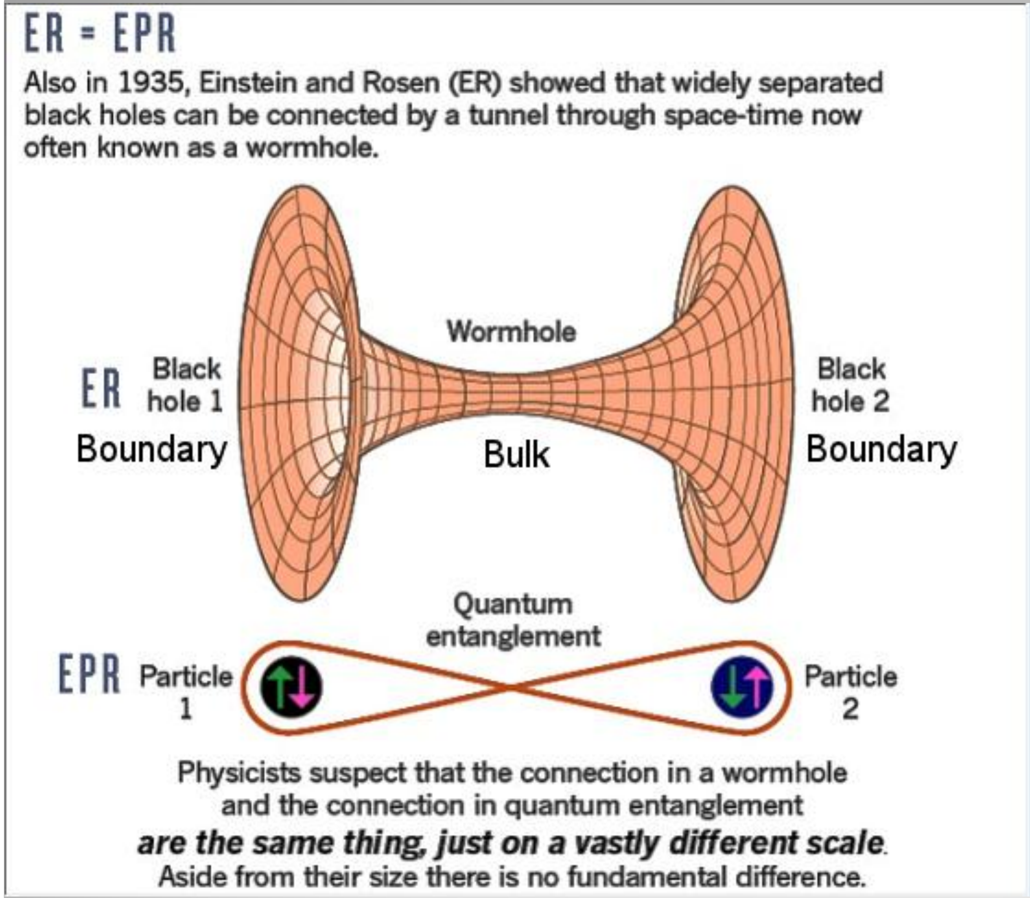
Conformal field theory, strong coupling condensed matter system, QCD,…

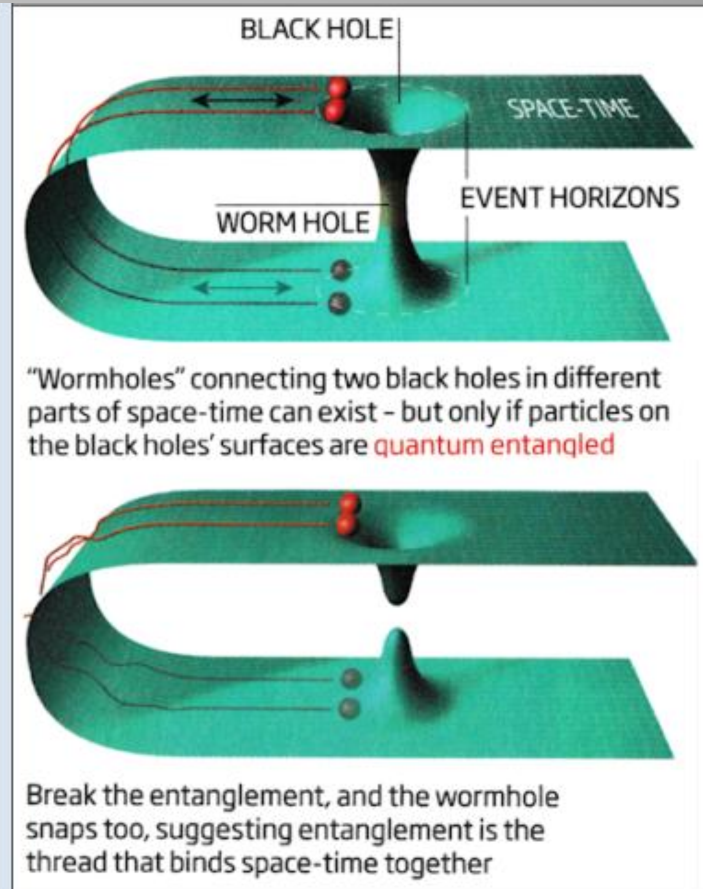
- Quantum information theory is a good lens to consider AdS/CFT or the quantum gravity

- The holographic entanglement entropy (HEE) relates the EE to the area of minimal surfaces [hep-th/0603001]

$$S_A^{(\mathrm{Hol})} = \underset{\substack{\gamma_A \\ \partial\gamma_A \sim \partial A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_\mathrm{A})}{4G_N} \right].$$

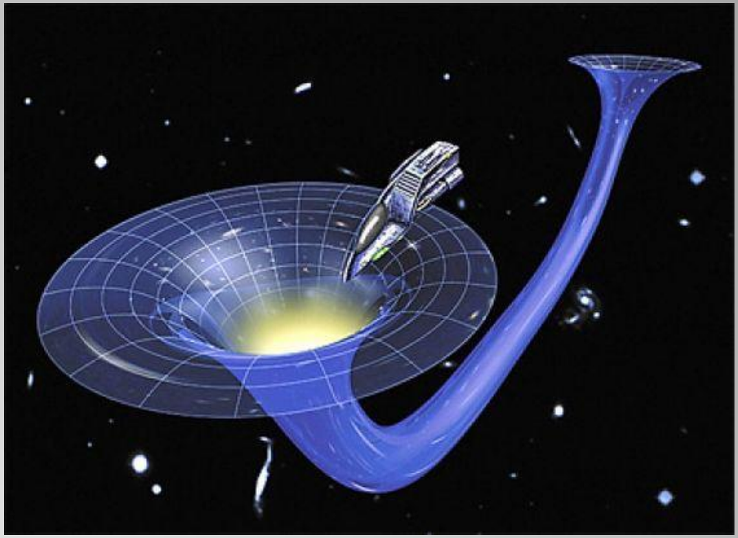



ER = EPR
Also in 1935, Einstein and Rosen (ER) showed that widely separated black holes can be connected by a tunnel through space-time now often known as a wormhole.
Wormhole
ER
Black hole 1
Boundary
Bulk
Black hole 2
Boundary
Quantum entanglement
EPR
Particle 1
Particle 2
Physicists suspect that the connection in a wormhole and the connection in quantum entanglement
are the same thing, just on a vastly different scale.
Aside from their size there is no fundamental difference.


BLACK HOLE
SPACE-TIME
EVENT HORIZONS
WORM HOLE
"Wormholes" connecting two black holes in different parts of space-time can exist – but only if particles on the black holes' surfaces are quantum entangled
Break the entanglement, and the wormhole snaps too, suggesting entanglement is the thread that binds space-time together

ER=EPR

- Once we accept that entanglement creates Einstein-Rosen bridges, then it becomes possible for to produce particles that come through the ERB, and arrive other side.



- What's the condition and how difficult it is?

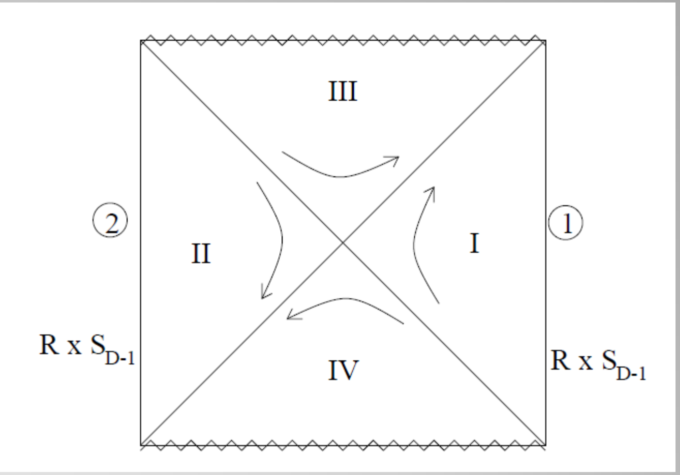
The concept named complexity is needed.

ER=EPR isn't enough!

Complexity of a TFD state

- An external AdS black hole is dual to a thermofield double (TFD) state,

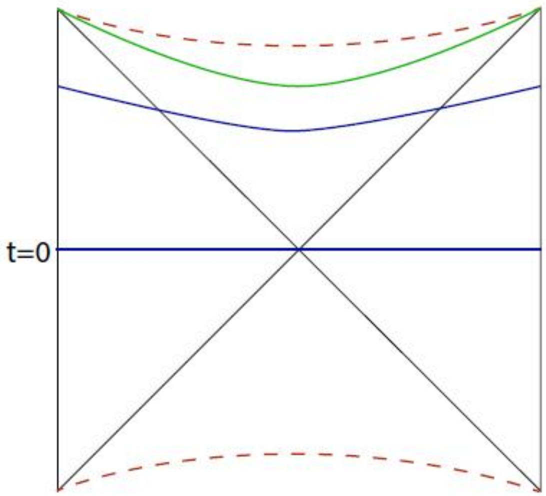
$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |E_n\rangle_1 \times |E_n\rangle_2$$

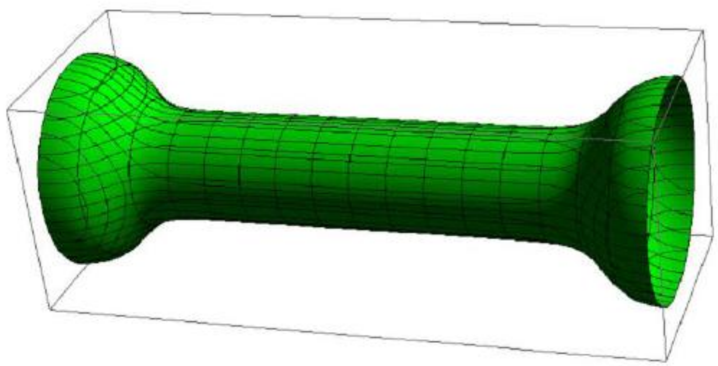


- What is the complexity of this TFD states?

- The first conjecture states that the complexity is given by the volume of the codimension-one maximal bulk surface that ends on the boundary at a time t

$$\mathcal{C} = \frac{V}{G\ell}$$

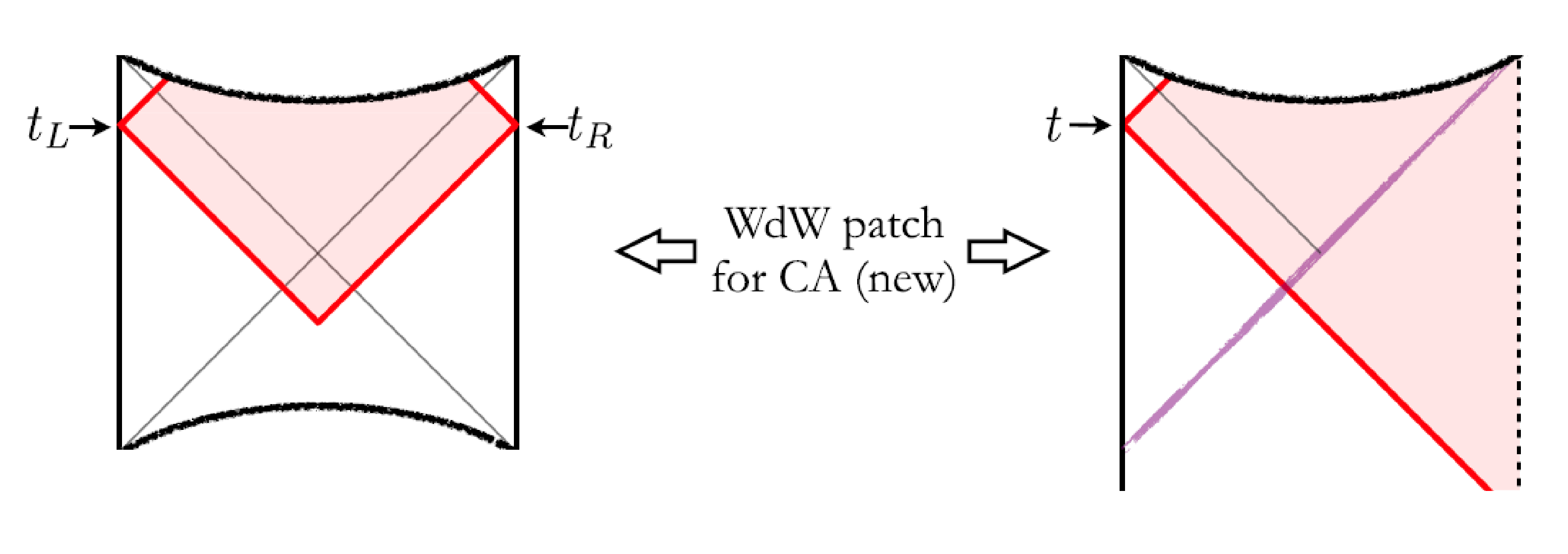




CV conjecture

- The second conjecture states that the complexity is given by the bulk action evaluated on the Wheeler-deWitt patch attached at some boundary time t

$$\mathcal{C} = \frac{\mathcal{A}}{\pi \hbar}$$



CA conjecture

- For vacuum space-time, the action in a sub-region is

$$I = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(R+\frac{d(d-1)}{L^2}\right) + \frac{1}{8\pi G_N}\int_{\mathcal{B}} d^d x\sqrt{|h|}\,K$$
$$- \frac{1}{8\pi G_N}\int_{\mathcal{B}'} d\lambda\, d^{d-1}\theta\sqrt{\gamma}\,\kappa + \frac{1}{8\pi G_N}\int_{\Sigma} d^{d-1}x\sqrt{\sigma}\,\eta + \frac{1}{8\pi G_N}\int_{\Sigma'} d^{d-1}x\sqrt{\sigma}\,a\,.$$

Hilbert-Einstein action

Gibbons-Hawking-York extrinsic curvature term

Null boundary terms

Hayward joint terms (corner terms)

Null joint terms (corner terms)

Full action in a sub-region

Susskind said in “Computational complexity and black hole horizons ”(《Fortschritte Der Physik》 , 2016 , 64 (1) :24 -43)

4.2 Gravity and Complexity

Entropic theories of gravity [29][30] build in various ways on the parallels between general relativity and thermodynamics, and on entropic forces in statistical mechanics. Undoubtedly there is truth to these ideas. I want to suggest that there may be another deep connection; this time between gravity and complexity. To state it as a slogan:

Things fall because there is a tendency toward complexity.

PHYSICAL REVIEW LETTERS **120,** 031601 (2018)

Einstein Equations from Varying Complexity

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(Received 4 September 2017; published 18 January 2018)

A recent proposal equates the circuit complexity of a quantum gravity state with the gravitational action of a certain patch of spacetime. Since Einstein's equations follow from varying the action, it should be possible to derive them by varying complexity. I present such a derivation for vacuum solutions of pure Einstein gravity in three-dimensional asymptotically anti–de Sitter space. The argument relies on known facts about holography and on properties of tensor network renormalization, an algorithm for coarse-graining (and optimizing) tensor networks.

DOI: 10.1103/PhysRevLett.120.031601

Introduction.—The AdS/CFT correspondence (holographic duality) [1] is the most powerful known approach to quantum gravity. It posits that every physical quantity in

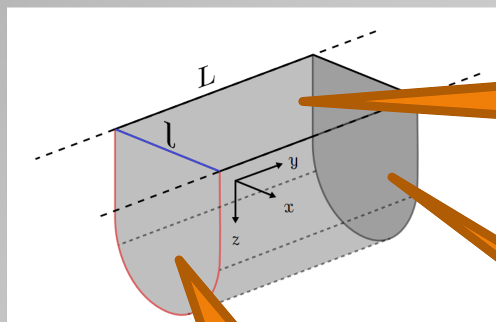
independently known facts about AdS gravity. The present Letter therefore provides a novel check of the $\mathcal{A} \propto \mathcal{C}$ conjecture. However, because circuit complexity is cur-

arXiv:1706.00965v1

Obtain the RG by complexity

Subregion complexity

- The CV subregions complexity was also proposed:
-



Boundary region

RT surface

Volume of extreme surface

- In 2016-2017, there are more than 100 papers involved the complexity in holography and black holes!

- However, some foundations about complexity are still unclear!

- What is the meaning of complexity in continuous systems?
- How do we give the complexity a well-defined mathematical foundation?
- What is the meaning of complexity in quantum field theory?
- …….

Complexity: the foundation is still absent!

Path Integral Complexity

- The basic tool to "define/compute" wave functions in QFT is the Euclidean PI

$$\Psi[\varphi] = \int_{\varphi(0,x)=\varphi_0(x)} D[\varphi]\exp[-S_E]$$

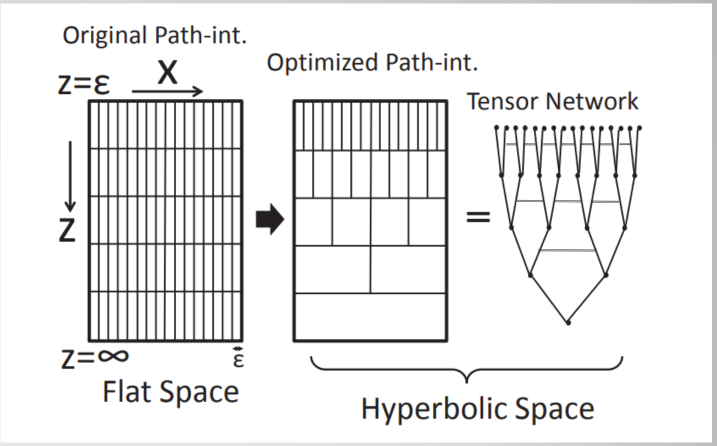
How can we optimize it and/or quantify its complexity?

For 2-D CFT with Liouville

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz \left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu e^{2\phi}\right]$$

Curvature (~Number of Isometries [Czech'17])

Volume (~Number of tensors)



Complexity between ground state and field eigenstate $C = S_E$

- Why the on-shell Euclidean action is a kind of complexity?

- How do we use it to compute the complexity between arbitrary pure states?

- How do we use it to compute the complexity between mixed states?

The problems

Fubini-Study metric and states complexity

- The basic idea is that the complexity between states should be a kind of distance;
- The states in a Hilbert space form a CP^n sphere;
- The only Riemannian metric preventing the symmetry in CP^n sphere is Fubini-Study metric;

$$ds^2 = \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle d\psi | \psi \rangle}{\langle \psi | \psi \rangle} \frac{\langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle}$$

- The complexity between two states then is the geodesic length connecting them

$$C\left(|\psi_1\rangle,|\psi_2\rangle\right)=\arccos\left|\langle\psi_1|\psi_2\rangle\right|$$

- Consider TFD states in CFTs

$$|\psi(T)\rangle=\prod_k\frac{1}{Z(\omega_k,T)}\left(\sum_{n=0}^{\infty}e^{-\frac{n\omega_k}{2T}}|n,k\rangle_L|n,k\rangle_R\right)\Rightarrow\left|\langle\psi(0)|\psi(T)\rangle\right|=\exp\left(-VT^{d-1}\right)\to 0$$

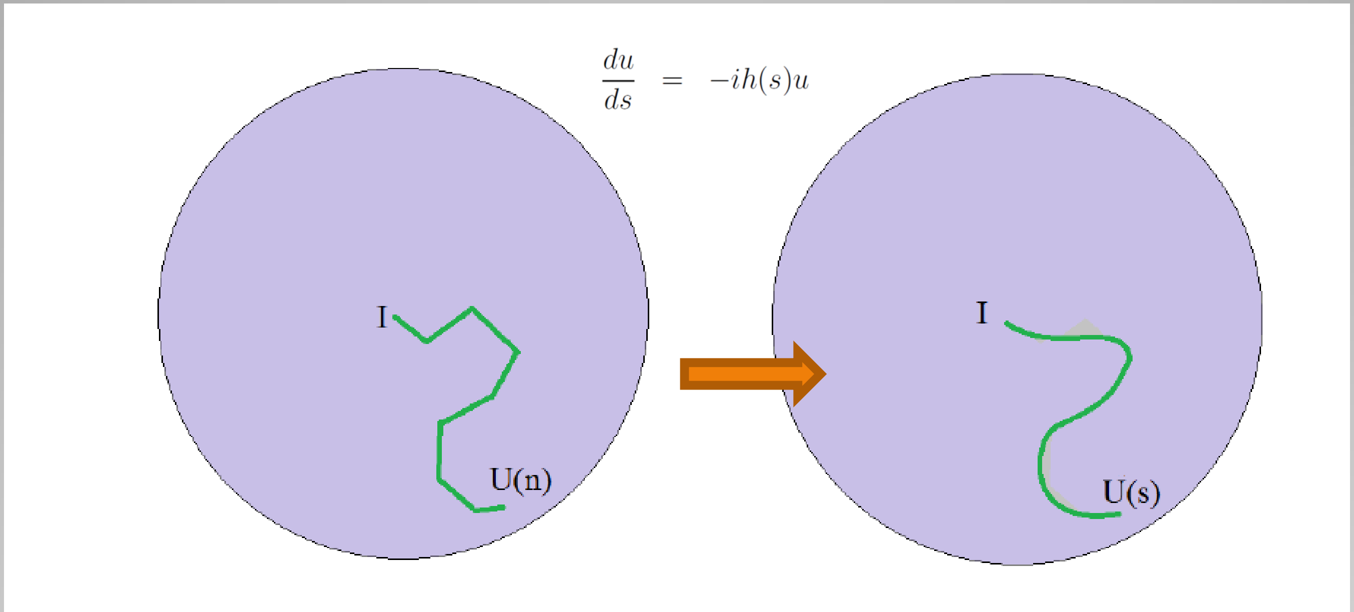
Trivial result! $C\left(|\psi(0)\rangle,|\psi(T)\rangle\right)=\frac{\pi}{2}$

- In order to obtain nontrivial result, the transformation between states can only be SU(1,1) operator.

- But why do we have to use SU(1,1) operators? What is the relationship to path integral complexity?

The problems

The minimal number of gates → The minimal length of the curve connecting I and U



The left side shows a discrete path induced by a series of gates. The right side shows a curve induced by a tangent operator h(s).

How to define the length of a curve in a operators set?

Finsler geometry and operator complexity

- For any piecewise C^1 curve $c : [0; 1] \to U$ which satisfies $c(0) = I$ and $c(1) = U$, one can define its length such that,

$$L[c] := \int_0^1 F\left[c(t), \frac{d}{dt}c(t)\right] dt$$

- The complexity of an operator U, $C(U)$, then is defined by

$$\min\{L[c] | \forall c: [0,1] \mapsto U, \exists \lambda \neq 0, s.t., c(0) = I\ , c(1) = \lambda U\}$$

- Here I is the identity of U.

Define complexity by Finsler geometry

M. A. Nielsen, M. R. Dowling, M. Gu and A. C. Doherty, Quantum computation as geometry, Science 311 (2006) 1133-1135

- It cannot compute the complexity between two states;
- The Finsler metric cannot be given uniquely;
- It is not clear about what is the relationship between it and other two complexity proposals.

The problem

Axiomatization of complexity

Based on **arXiv: 1803.01797**

Run-Qiu Yang, Yu-Sen An, Chao Niu, Cheng-Yong Zhang and Keun-Young Kim

Basic properties of complexity

- The complexity for any operator should be nonnegative;

$$C(\hat{O}) \geq 0, \forall \hat{O} \in D$$

- As the identity can be realized without referring any gate, its complexity should be zero; if an operator has zero complexity, then it can be realized by a quantum circuit without any gate and so this operator must be identity.

$$C(\hat{O}) = 0 \Leftrightarrow \hat{O} = \hat{I}$$

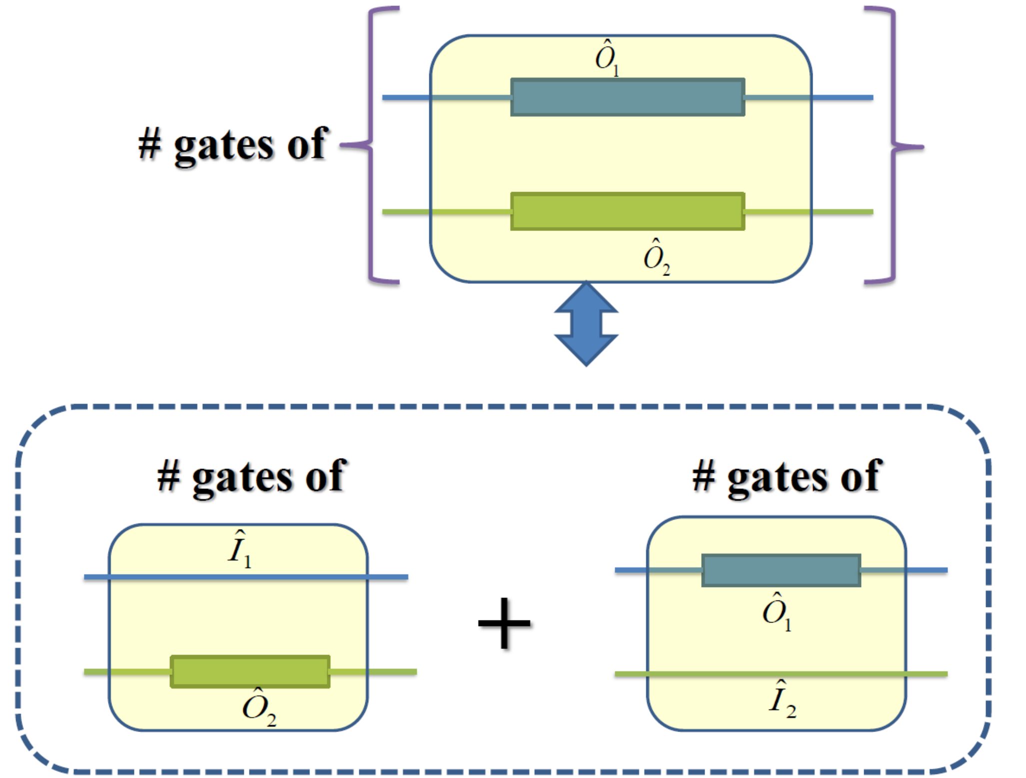
- As the complexity should stand for the minimal required gates, following inequality must be true,

$$C(\hat{O}_1 \hat{O}_2) \leq C(\hat{O}_1) + C(\hat{O}_2), \forall \hat{O}_1, \hat{O}_2 \in D$$

Cartesian product

The fourth axiom :

$$C[(\hat{O}_1, \hat{O}_2)] = C[(\hat{O}_1, \hat{I}_2)] + C[(\hat{I}_1, \hat{O}_2)]$$



- As any quantum circuit can only realize a unitary transformation, it is enough to only consider the complexity for operators in SU(n) groups.

- **Assumption 1**: for any generator H, the complexity of exp($H\varepsilon$) depends on ε smoothly if $\varepsilon \geq 0$;

- **Assumption 2**: complexity has path-reversal symmetry.

Two assumptions for SU(n) grups

Complexity and Finsler geometry

- In fact, just by four axioms and two assumptions, we can prove following surprising result:

The complexity for SU(n) group can **only** be given by geodesic length of a **bi-invariant Finsler geometry** in SU(n) group with following Finsler structure:

$$F(c,\dot{c}) = \lambda \mathrm{Tr}\left(\sqrt{HH^\dagger}\right) = \lambda \mathrm{Tr}\left(\sqrt{\dot{c}\dot{c}^\dagger}\right)$$

Here H is defined as $H = \dot{c}c^{-1}$ or $c^{-1}\dot{c}$

In general it is not Riemannian

- In fact, it is enough for any quantum mechanics and quantum field theoeries.

Applications

- The complexity between two states is given by density matrixes such that:

$$\mathcal{C}(\rho_1,\rho_2) = -2\ln\left\{\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_2}}\right\}$$

- For two pure states $\rho_i = |\psi_i\rangle\langle\psi_1|$, we can prove this is just

$$\mathcal{C}(\rho_1,\rho_2) = -\ln\left|\langle\psi_1|\psi_2\rangle\right|^2$$

- In addition, if one of them is ground state and the other is field eigenstate, it can be expressed as Euclidean path integral:

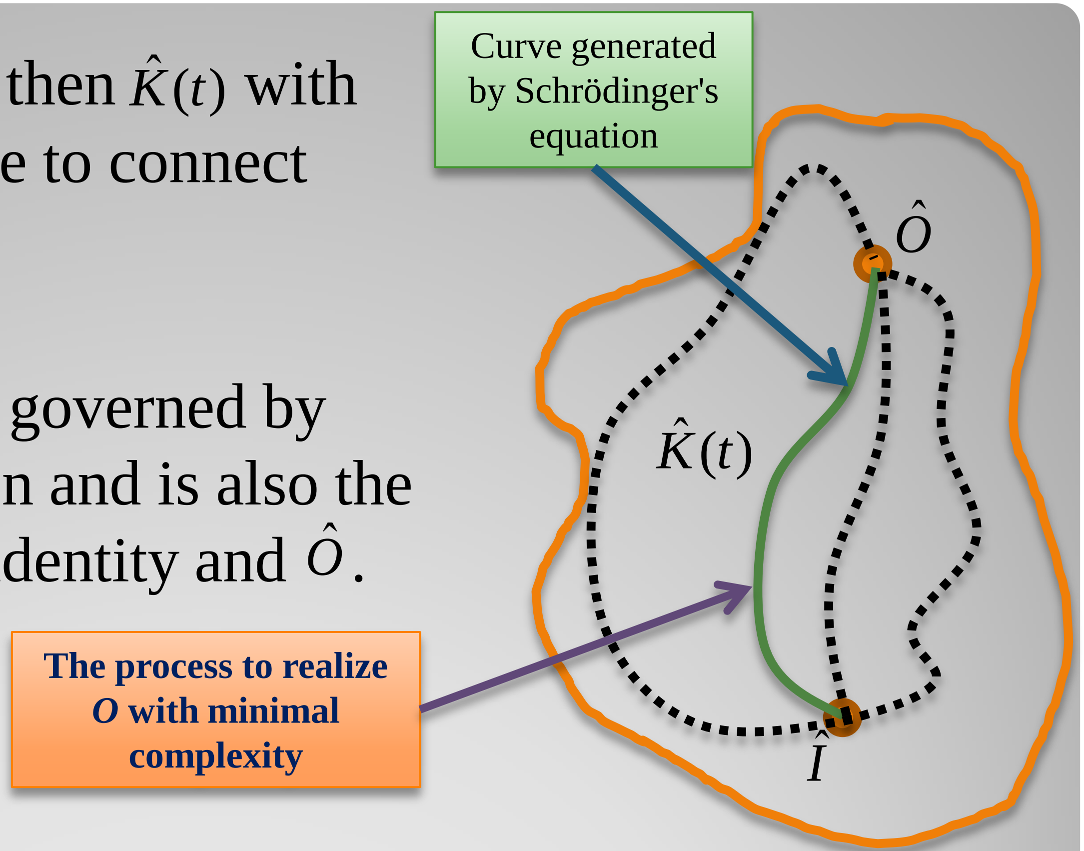
$$\mathcal{C}(\rho_1,\rho_2) = -\ln\int_{\phi(x,0)=\phi_0(x)} D[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)$$

- For classical limit, its leading term reads

$$\mathcal{C}(\rho_1,\rho_2) \approx \frac{1}{\hbar}\min\left\{S_E[\phi]-S_E[0]\,|\,\forall\phi(x,\tau),\ \text{s.t.,}\ \phi(x,0)=\phi_0(x)\right\}$$

- Assume that $\hat{K}(t_0) = \hat{O}$, then $\hat{K}(t)$ with $t \in [0, t_0]$ forms a curve to connect identity and $\hat{O}$.

- The physical curve is governed by Schrödinger's equation and is also the geodesic connecting identity and $\hat{O}$.



For an isolated system, the time evolution operator will go along the curve such that the complexity in this process is locally minimal!

Schrödinger's Equation is the geodesic equation

Minimal complexity principle?

- We have three equivalent formulations about the dynamic of quantum mechanics when Hamilton is time-independent:
- **Schrödinger's equation:** the time-evolution operator satisfies following equation

$$\frac{\partial}{\partial t}\hat{K}(t) = -i\hbar^{-1}\hat{H}\hat{K}(t), \quad \hat{K}(0) = \hat{I}$$

- **Path integral:** the time-evolution operator at the coordinate representation is called propagator and is given by

$$\langle x_1 | \hat{K}(t) | x_2 \rangle = K(x_1, x_2; t) = \int_{x(0)=x_2}^{x(t)=x_1} \mathcal{D}[x] \exp\left[i\hbar^{-1} \int_0^t L(x, \dot{x}) \mathrm{d}t \right]$$

- **Minimal complexity principle:** in isolated system time-evolution operator is given so that its corresponding process has minimal complexity.

- Complexity is a quantity to describe amount of computation in a task;
- It has some holographic proposals and been studied well in holography.
- Some attempts have been done to give it a well definition in quantum field theories;
- A well and complete definition on complexity is still not clear.

Summary