An alternative parametrization for neutrino mixing

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Neutrinos

- Abundance in the Universe
- Member of the quark-lepton spectra of fundamental particles
- Indispensible in electroweak interactions
- Properties:
  - extrinsic properties: well-established in SM
  - intrinsic properties: not so well-known
- After decades of effort
  $\Rightarrow$ ν_s change flavor after propagating a finite distance in vaccum and in matter
  $\Rightarrow$ neutrino oscillation

Neutrino Oscillation and Mixing

- Consistent explanation to ν oscillation is to postulate:
  - non-zero, distinctive neutrino masses
  - neutrino mixing

  $\Rightarrow$ weak eigenstates $\neq$ mass eigenstates
- ν mass: mass eigenstates ν_i with mass m_i
- ν mixing: consider, *e.g.*, $W^+ \rightarrow \nu_i + \bar{l}_\alpha$
  - accompanying ν eigenstate is not always the same ν_i
  - $V^*_{\alpha i}$ = amplitude for producing a specific pair of $\bar{l}_\alpha + \nu_i$

  $\Rightarrow$ neutrino of flavor α emitted in W^+ decay is a superposition of mass eigenstates: $\nu_\alpha = \sum_i V^*_{\alpha i} \nu_i$

Mixing Matrix

- $V_{\alpha i}$ can be collected into the mixing matrix V (or V_{PMNS}, Pontecorvo-Maki-Nakagawa-Sakata)
- $V^\dagger (MM^\dagger) V = Diag(m_1^2, m_2^2, m_3^2)$
  M: neutrino mass matrix
- unitarity of V guarantees that ν_α goes with l_α in interaction
- inverted relation: $\nu_i = \sum_\alpha V_{\alpha i} \nu_\alpha$
- $|V_{\alpha i}|^2 =$ fraction of ν_α in ν_i
- $|V_{\alpha i}|^2 =$ prob. that the charged lepton will be of flavor α as ν_i interacts

ν Transition Probability

- ν oscillation: ν at birth as ν_α, but detected as ν_β
- If $\beta \neq \alpha$, the neutrino has morphed from ν_α to ν_β
- $i\frac{d\nu_j}{dt} = H\nu_j$
- $A(\nu_\alpha \to \nu_\beta) = \sum_j V_{\alpha j} \exp(-i\frac{m_j^2 t}{2E}) V_{j\beta}^\dagger$
- $P(\nu_\alpha \to \nu_\beta) =$
  $\delta_{\alpha\beta} - 4\sum_{i>j} Re[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*] \sin^2(\Delta m_{ij}^2 \frac{L}{4E})$
  $+2Im[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*] \sin(\Delta m_{ij}^2 \frac{L}{2E})$,

  with $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ (in vacuum)

Parametrization for Mixing Matrix

- "Standard" Parametrization of V (Particle Data Group):
  - Three mixing angles θ_{12}, θ_{23}, θ_{13}
  - One CP-odd phase ϕ
  - $V = V(\theta_{12}, \theta_{23}, \theta_{13}, \phi)$
- $V(\theta_{12}, \theta_{23}, \theta_{13}, \phi)$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\phi} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\phi} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\phi} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\phi} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\phi} & c_{13}c_{23} \end{pmatrix}$$

with $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$.

Current data from Oscillation

- $\Delta m^2_{21} \sim 7 \times 10^{-5} eV^2$, $\sin^2\theta_{12} \sim 0.3$
- $|\Delta m^2_{32}| \sim 2.4 \times 10^{-3} eV^2$, $\sin^2\theta_{23} \sim 0.5$
- $\sin^2\theta_{13} < 0.03$
  $\Rightarrow \Delta m^2_{21}/|\Delta m^2_{32}| \sim 0.03$
- $\Delta m^2_{32} > 0$, normal hierarchy
  $\Delta m^2_{32} < 0$, inverted hierarchy
- ϕ so far unconstrained

Possible Mass Hierarchies

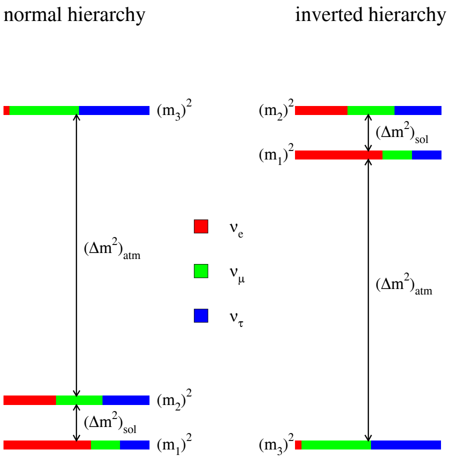
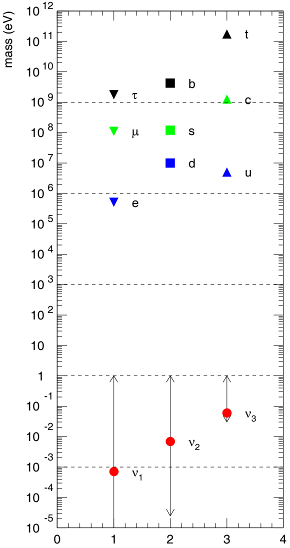


Figure:

Fermion Masses



Fermion Mixing

Patterns of quark and ν mixing

$$|V_{CKM}| \sim \begin{pmatrix} 1 & 0.2 & 0.001 \\ 0.2 & 1 & 0.01 \\ 0.001 & 0.01 & 1 \end{pmatrix}$$

$$|V_{PMNS}| \sim \begin{pmatrix} 0.8 & 0.5 & < 0.2 \\ 0.4 & 0.6 & 0.7 \\ 0.4 & 0.6 & 0.7 \end{pmatrix}$$

$\Rightarrow$ They look very different!

- What, if any, is the relationship between V_{CKM} and V_{PMNS}?
- What, if any, is the principle responsible for the observed ν masses and leptonic mixing?

Important Questions in ν physics

- Absolute mass scale?
- Is CP phase non-zero?
- Is θ_{23} exactly maximal?
- Sign of Δm^2_{32} (mass hierarchy)?
- How small is θ_{13}?

Future Oscillation Experiments

- Focus of future oscillation experiments:
  Better determinations of elements of neutrino mass and mixing matrtices.
- Neutrino oscillation exp. with better precision
  - θ_{12}, θ_{23}, θ_{13}
  - Δm^2_{21}, magnitude and sign of $|\Delta m^2_{32}|$
  - Dirac phase ϕ: long-baseline $\nu_\mu \to \nu_e$ oscillation

$2-\nu$ Oscillation in Vacuum

- $P(\nu_e \rightarrow \nu_e) = |<\nu_e|\nu_e>|^2 = 1 - \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$
  - $\Delta m^2 = m_2^2 - m_1^2$
  - $L=$ source-to-detector distance
- $\sin^2[\Delta m^2 \frac{L}{4E}] \simeq \sin^2[1.27(\Delta m^2/eV^2)(\frac{L/km}{E/GeV})]$

  $\Rightarrow$ experiment sensitive to $\Delta m^2 \sim (\frac{L/km}{E/GeV})^{-1} eV^2$

$2-\nu$ Oscillation in Uniform Medium

- Coherent forward scattering:
  - W exchange between ν_e and e
  - Z exchange between $\nu_{e,\mu,\tau}$ and e, p, n
- Extra interacting potential energy for ν_e from CC
- $A = 2\sqrt{2} G_F N_e E$ (a measure of matter effect)
- $\theta \longrightarrow \theta_M$, $\nu_i \longrightarrow \nu_{iM}$
- $P_M(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta_M \sin^2(\Delta m^2_M \frac{L}{4E})$
  - $\Delta m^2_M \equiv \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - A/\Delta m^2)^2}$
  - $\sin^2 2\theta_M \equiv \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - A/\Delta m^2)^2}$

Medium with Varying Density

- Examples: solar or SN neutrinos

$$i\frac{d}{dx}\begin{pmatrix}\nu_{1M}\\ \nu_{2M}\end{pmatrix} = \begin{pmatrix}-\Delta m^2/4E & -id\theta_M/dx\\ id\theta_M/dx & \Delta m^2/4E\end{pmatrix}\begin{pmatrix}\nu_{1M}\\ \nu_{2M}\end{pmatrix}$$

- If $|d\theta_M/dx| \ll |\Delta m^2/4E|$ (adiabatic approximation)

  $\Rightarrow \nu_{1M} \leftrightarrow \nu_{2M}$ transition suppressed

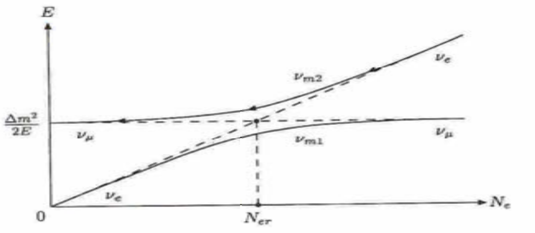
- $\frac{d\theta_M}{dx} = \frac{1}{2}\frac{\sin 2\theta(2\sqrt{2}G_F E)}{(\frac{A}{\Delta m^2}-\cos 2\theta)^2+\sin^2 2\theta}\left(\frac{dN_e}{dx}\right)$

- At resonance ($A = \Delta m^2 \cos 2\theta$), adiabaticity is maximally violated:

  $\frac{d\theta_M}{dx} \rightarrow max.$, $\frac{\Delta m^2}{4E} \rightarrow min.$

  $\Rightarrow$ non-vanishing prob. for $\nu_{1M} \leftrightarrow \nu_{2M}$ at resonance

$2 - \nu$ Resonant Transition



$3 - \nu$ Parameters

- Parameters for ν oscillation:

  θ_{12}, θ_{23}, θ_{13}, $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$,
  $|\Delta m_{32}^2| \equiv |m_3^2 - m_2^2| \simeq |m_3^2 - m_1^2|$
- Unknown Ocsillation Parameters
  - sign of Δm_{32}^2: ambiguity in neutrino mass hierarchy
    normal hierarchy: $m_3 \gg m_2, m_1$
    inverted hierarchy: $m_3 \ll m_2, m_1$
  - mixing angle θ_{13} (current upper bound $\theta_{13} \sim 10^{-2}$)

Neutrino Mass Hierarchies

- normal hierarchy: both higher and lower transition occur in ν sector
- inverted hierarchy: higher transition in $\bar{\nu}$ sector, while lower transition in ν sector

Resonant Transitions

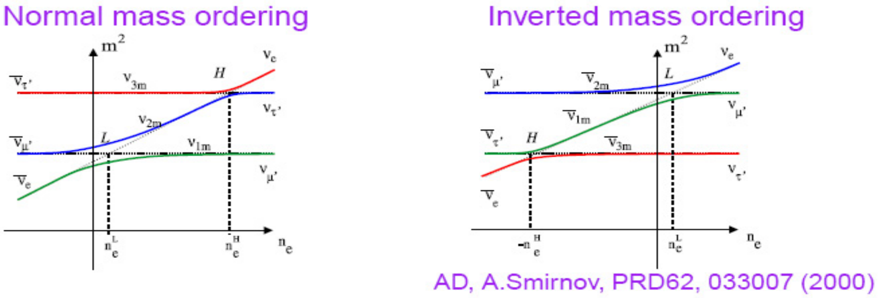


Figure:

Why another parametrization?

Mixing and oscillation are modified by matter:

- matter effects can fake CP effects in long baseline exp.
- solar and supernova neutrinos spectra are modified before detection

One needs to know

- how mixing parameters evolve in matter
- how the mixing parameters in vacuum are related to that in vacuum

- In the literature, usually the cubic eigenvalue problems are involved
  - Calculations are formidable
  - Results are far from transparent for clear extraction of physical implications
  - Intriguing properties and symmetries may lost
- For more general study of mixing problems

Alternative parametrization for neutrino mixing

- Consider the mixing matrix that satisfies detV=+1
- A set of six rephasing invariants can be constructed: $\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}$, where (i, j, k) is a permutation of $(1, 2, 3)$
- We can show that $\Gamma_{ijk} = R_{ijk} - iJ$
- Even and odd permutations of the real part R_{ijk}:
  $(R_{123}, R_{231}, R_{312}) = (x_1, x_2, x_3)$
  $(R_{132}, R_{213}, R_{321}) = (y_1, y_2, y_3)$
- Imaginary part: J=Jarlskog invariant (a measure of CP effect)

Jarlskog invariant J

- $Im[V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}] \equiv J(\sum_\gamma \varepsilon_{\alpha\beta\gamma} \sum_k \varepsilon_{ijk})$
- in $x - y$ parametrization: $J^2 = x_1 x_2 x_3 - y_1 y_2 y_3$
- in standard parametrization:
  $J = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\phi$
- CP and T-violating asymmetries:
  - $\Delta P_{CP} = P(\nu_a \to \nu_b) - P(\bar{\nu}_a \to \bar{\nu}_b)$
  - $\Delta P_T = P(\nu_a \to \nu_b) - P(\nu_b \to \nu_a)$
- $\Delta P_{CP} = 16 J \sin(\frac{M^2_{12} L}{2}) \sin(\frac{M^2_{23} L}{2}) \sin(\frac{M^2_{31} L}{2})$ + matter term
- In vacuum, $\Delta P_{CP} = \Delta P_T$ (matter term=0 for ΔP_T)
- Any indication of $\Delta P_{CP} \neq \Delta P_T$ in matter would help extracting J.

Constraints and Properties

- Two constraints:
  $detV = (x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1$
  $(x_1x_2 + x_2x_3 + x_3x_1) - (y_1y_2 + y_2y_3 + y_3y_1) = 0$
- squared mixing elements $|V_{ij}|^2$:

$$|V_{ij}|^2 = \begin{pmatrix} |V_{11}|^2 & |V_{12}|^2 & |V_{13}|^2 \\ |V_{21}|^2 & |V_{22}|^2 & |V_{23}|^2 \\ |V_{31}|^2 & |V_{32}|^2 & |V_{33}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix}$$

Effective Hamiltonian in matter

In flavor basis

- $H_{eff} = \frac{1}{2E}\left[V\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}V^\dagger + \begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix}\right] \equiv \frac{H}{2E}$
  where $A = \sqrt{2}G_F n_e E$
- H is diagonalized by the mixing matrix U in matter:
  $$H = UDU^\dagger = U\begin{pmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \end{pmatrix}U^\dagger$$
  with $D_i = M_i^2$ (effective squared masses in matter)

Evolution Equations

- We may start with

$$\frac{dH}{dA} = \frac{d}{dA}[UDU^\dagger] = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

- $U^\dagger \frac{dU}{dA} D + \frac{dD}{dA} + D \frac{dU^\dagger}{dA} U = U^\dagger \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} U$

$$= \begin{pmatrix} |U_{11}|^2 & U_{12} U_{11}^* & U_{13} U_{11}^* \\ U_{11} U_{12}^* & |U_{12}|^2 & U_{13} U_{12}^* \\ U_{11} U_{13}^* & U_{12} U_{13}^* & |U_{13}|^2 \end{pmatrix}$$

- Diagonal and off-diagonal elements leads to the desired relations.

- Off-diagonal elements for $d(U^\dagger U)/dA$ vanish:
  $\left[U^\dagger \frac{dU}{dA}\right]_{ik} + \left[\frac{dU^\dagger}{dA} U\right]_{ik} = 0, (i \neq k)$
  $\Rightarrow (D_i - D_k) \left[\frac{dU^\dagger}{dA} U\right]_{ik} = U_{1k} U^*_{1i}$
- Since $\left[\frac{dU^\dagger}{dA} U\right]_{ik} \left[U^\dagger\right]_{kj} = \left[\frac{dU^\dagger}{dA}\right]_{ij}$,
  we get
  - $\left[\frac{dU^\dagger}{dA}\right]_{ij} = \frac{dU^*_{ji}}{dA} = \sum_{k\neq i} \frac{U_{1k} U^*_{1i}}{D_i - D_k} U^*_{jk}$,
  - $\frac{dU_{ij}}{dA} = \sum_{k\neq j} \frac{U_{ik} U_{1j}}{D_j - D_k} U^*_{1k}$

- In matter, $\Gamma_{ijk} = U_{1i}U_{2j}U_{3k} = R_{ijk} - iJ$
- $\frac{d\Gamma_{123}}{dA} = \frac{dx_1}{dA} - i\frac{dJ}{dA}$
  $= \frac{dU_{11}}{dA}U_{22}U_{33} + U_{11}\frac{dU_{22}}{dA}U_{33} + U_{11}U_{22}\frac{dU_{33}}{dA}$
- The real and imaginary parts:
  $\frac{dx_1}{dA} = \frac{x_1x_2 - 2x_1y_2 + y_1y_2}{D_1 - D_2} + \frac{-x_1x_2 + x_1x_3 + y_1y_2 - y_1y_3}{D_2 - D_3} + \frac{-x_1x_3 + 2x_1y_3 - y_1y_3}{D_3 - D_1}$
  $\frac{1}{J}\frac{dJ}{dA} = \frac{d\ln J}{dA} = \frac{-x_1 + x_2 + y_1 - y_2}{D_1 - D_2} + \frac{-x_2 + x_3 + y_2 - y_3}{D_2 - D_3} + \frac{x_1 - x_3 - y_1 + y_3}{D_3 - D_1}$
- Use other permutations of $d\Gamma_{ijk}/dA$ to get the rest:

- The off-diagonal elements lead to

$$\frac{dx_1}{dA} = \frac{x_1x_2-2x_1y_2+y_1y_2}{D_1-D_2} + \frac{-x_1x_2+x_1x_3+y_1y_2-y_1y_3}{D_2-D_3} + \frac{-x_1x_3+2x_1y_3-y_1y_3}{D_3-D_1}$$
$$\frac{dx_2}{dA} = \frac{-x_1x_2+2x_2y_1-y_1y_2}{D_1-D_2} + \frac{x_2x_3-2x_2y_3+y_2y_3}{D_2-D_3} + \frac{x_1x_2-x_2x_3-y_1y_2+y_2y_3}{D_3-D_1}$$
$$\frac{dx_3}{dA} = \frac{-x_1x_3+x_2x_3+y_1y_3-y_2y_3}{D_1-D_2} + \frac{-x_2x_3+2x_3y_2-y_2y_3}{D_2-D_3} + \frac{x_1x_3-2x_3y_1+y_1y_3}{D_3-D_1}$$
$$\frac{dy_1}{dA} = \frac{-x_1x_2+2x_2y_1-y_1y_2}{D_1-D_2} + \frac{-x_1x_2+x_1x_3+y_1y_2-y_1y_3}{D_2-D_3} + \frac{x_1x_3-2x_3y_1+y_1y_3}{D_3-D_1}$$
$$\frac{dy_2}{dA} = \frac{x_1x_2-2x_1y_2+y_1y_2}{D_1-D_2} + \frac{-x_2x_3+2x_3y_2-y_2y_3}{D_2-D_3} + \frac{x_1x_2-x_2x_3-y_1y_2+y_2y_3}{D_3-D_1}$$
$$\frac{dy_3}{dA} = \frac{-x_1x_3+x_2x_3+y_1y_3-y_2y_3}{D_1-D_2} + \frac{x_2x_3-2x_2y_3+y_2y_3}{D_2-D_3} + \frac{-x_1x_3+2x_1y_3-y_1y_3}{D_3-D_1}$$

- The diagonal elements lead to

$$\frac{dD_1}{dA} = |U_{11}|^2 = x_1 - y_1, \frac{dD_2}{dA} = |U_{12}|^2 = x_2 - y_2,$$
$$\frac{dD_3}{dA} = |U_{13}|^2 = x_3 - y_3$$

Initial Conditions

- Standard Parametrization

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\phi} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\phi} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\phi} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\phi} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\phi} & c_{13}c_{23} \end{pmatrix}$$

- Tri-bimaximal scenario:

$$|V_{ij}|^2 \sim \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix},$$

with $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$.

- $s_{12} = (1+\eta)/\sqrt{3}$, $s_{23} = (1+\xi)/\sqrt{2}$, and $s_{13} = \epsilon$
- Solving for initial conditions from

$$|V_{ij}(x,y)|^2 = |V_{ij}(\eta,\xi,\epsilon,\phi)|^2$$

- For simplicity, we take $\eta = \xi = 0$ and ignore terms in $O(\beta\epsilon^3)$, where $\beta = (\sqrt{2}/3)\cos\phi$.
- $x_{10} = \frac{1}{6}(2 - 3\beta\epsilon - 2\epsilon^2), \quad y_{10} = \frac{1}{6}(-2 - 3\beta\epsilon + 2\epsilon^2)$
  $x_{20} = \frac{1}{6}(1 - 3\beta\epsilon - \epsilon^2), \quad y_{20} = \frac{1}{6}(-1 - 3\beta\epsilon + \epsilon^2),$
  $x_{30} = \frac{1}{2}(\beta\epsilon + \epsilon^2), \quad y_{30} = \frac{1}{2}(\beta\epsilon - \epsilon^2)$

Notations and input parameters

- $\delta_0 = m_2^2 - m_1^2$, $\Delta_0 = |m_3^2 - m_2^2|$ in vacuum
  $\delta = M_2^2 - M_1^2$, $\Delta = |M_3^2 - M_2^2|$ in matter.
- $\delta_0 = 7.0 \times 10^{-5} eV^2$ and mass hierarchy $\delta_0/\Delta_0 = 1/32$
- $\epsilon = 0.17$ (current upper bound of $|V_{13}|$) and $\beta = 1/4$ (or $\phi \sim 58^\circ$)

Numerical Solutions

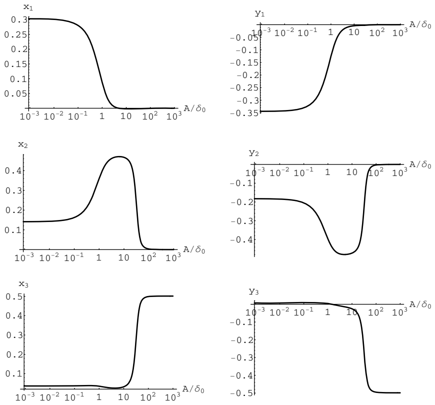


Figure: Note: $x_i + y_i \sim 0$

The following can be verified numerically:

- $(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1$
- $x_1 + x_2 + x_3 \simeq x_{10} + x_{20} + x_{30}$

  $y_1 + y_2 + y_3 \simeq y_{10} + y_{20} + y_{30}$

NOTE: $|\Sigma x_i - \Sigma x_{i0}|$ and $|\Sigma y_i - \Sigma y_{i0}|$ are of order $< |\beta\epsilon|$

Squared Matrix Elements

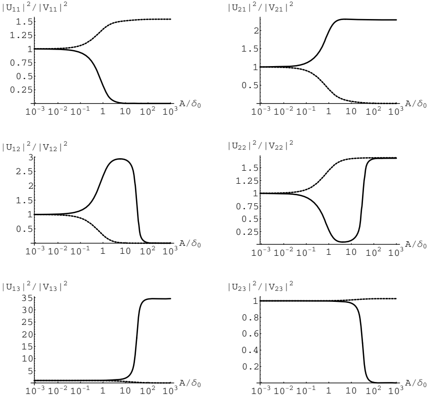


Figure: $|U_{ij}|^2/|V_{ij}|^2$ for ν (solid), and $\bar{\nu}$ (dashed, by replacing A with $-A$ and U with U^* in that of the ν sector.)

Jarlskog Invariant J^2

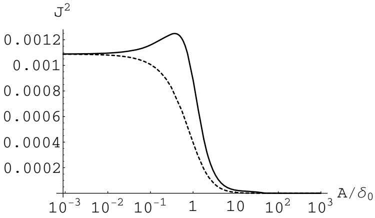


Figure: $J^2 = x_1 x_2 x_3 - y_1 y_2 y_3$

Mixing angles for Standard Parametrization

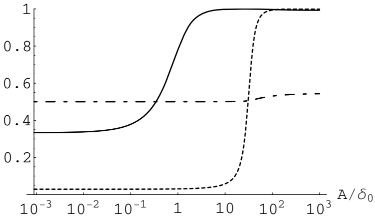


Figure: $\sin^2\theta_{12} = 1/(1+\frac{x_1-y_1}{x_2-y_2})(solid), \sin^2\theta_{23} = 1/(1+\frac{x_1-y_2}{x_2-y_1})$ (dot-dashed), and $\sin^2\theta_{13} = x_3 - y_3$ (dashed)

Approximation near the resonance

Clues from numerical solutions:

- (i) $x_1 + y_1 \sim 0$, $x_2 + y_2 \sim 0$, $x_3 + y_3 \sim 0$
- (ii) $|x_3| \ll 1$ and $|y_3| \ll 1$ near the lower resonance
- (iii) $|x_1| \ll 1$ and $|y_1| \ll 1$ near the higher resonance

Approximations using (i):

$$\frac{dx_1}{dA} \simeq -4x_1\left(\frac{x_2}{\delta} + \frac{x_3}{\Delta+\delta}\right) \simeq -\frac{dy_1}{dA}$$

$$\frac{dx_2}{dA} \simeq 4x_2\left(\frac{x_1}{\delta} - \frac{x_3}{\Delta}\right) \simeq -\frac{dy_2}{dA}$$

$$\frac{dx_3}{dA} \simeq 4x_3\left(\frac{x_2}{\Delta} + \frac{x_1}{\Delta+\delta}\right)$$

$$\frac{d\Delta}{dA} \simeq 2(x_3 - x_2), \qquad \frac{d\delta}{dA} \simeq 2(x_2 - x_1)$$

Near Lower Resonance

From (ii), $|x_3| \ll 1$ and $|y_3| \ll 1$ near the lower resonance:

$\frac{dx_1}{dA} \simeq \frac{-4x_1x_2}{\delta}, \quad \frac{dx_2}{dA} \simeq \frac{4x_1x_2}{\delta}, \quad \frac{dx_3}{dA} \to 0,$

$\frac{d\delta}{dA} \simeq 2(x_2 - x_1), \quad \frac{d\Delta}{dA} \simeq -2x_2.$

Relations to use, with $\gamma \equiv 1 - \beta\epsilon$:

- $x_1 + x_2 + x_3 \simeq x_{10} + x_{20} + x_{30} = \gamma/2$
- $x_1 \simeq (\gamma/2) - x_2 - x_{30}$
- $x_2 \simeq (\gamma/2) - x_1 - x_{30}$

Approximated solutions near lower resonance:

- $x_1 = (\frac{\gamma-2x_{30}}{4})\left[1 - \frac{(\gamma-2x_{30})A-\delta_0(\frac{4x_{10}}{\gamma-2x_{30}}-1)}{\delta}\right]$
- $x_2 = (\frac{\gamma-2x_{30}}{4})\left[1 + \frac{(\gamma-2x_{30})A-\delta_0(1-\frac{4x_{20}}{\gamma-2x_{30}})}{\delta}\right]$
- $x_3 = x_{30}$
- $\delta = \delta_0\sqrt{1 - 2\frac{A}{\delta_0}\left[4x_{10} - (\gamma - 2x_{30})\right] + (\gamma - 2x_{30})^2\frac{A^2}{\delta_0^2}}$

  $= \delta_0\sqrt{1 - 2\frac{A}{\delta_0}\left[(\gamma - 2x_{30}) - 4x_{20}\right] + (\gamma - 2x_{30})^2\frac{A^2}{\delta_0^2}}$
- $\Delta = \Delta_0 - (\frac{\gamma}{2} - x_{30})A + \frac{\delta_0}{2} - \frac{\delta}{2}$

Near Higher Resonance

From (iii), $|x_1| \ll 1$ near the higher resonance:

$\frac{dx_1}{dA} \to 0, \quad \frac{dx_2}{dA} \simeq -4x_2x_3/\Delta, \quad \frac{dx_3}{dA} \simeq 4x_2x_3/\Delta,$

$\frac{d\delta}{dA} \simeq 2x_2, \quad \frac{d\Delta}{dA} \simeq 2(x_3 - x_2)$

NOTE: "Projected" initial value for $x'_{20} = (\gamma/2) - x_{30}$

Approximated solutions near higher resonance:

- $x_1 = 0$
- $x_2 = \frac{\gamma}{4}\left[1 - \frac{\gamma A - \frac{\Delta_0}{\gamma}(4x'_{20} - \gamma)}{\Delta}\right]$
- $x_3 = \frac{\gamma}{4}\left[1 + \frac{\gamma A - \frac{\Delta_0}{\gamma}(\gamma - 4x_{30})}{\Delta}\right]$
- $\Delta = \Delta_0\sqrt{1 - 2\frac{A}{\Delta_0}(\gamma - 4x_{30}) + (\frac{\gamma A}{\Delta_0})^2}$
  $= \Delta_0\sqrt{1 - 2\frac{A}{\Delta_0}(4x'_{20} - \gamma) + (\frac{\gamma A}{\Delta_0})^2}$
- $\delta = \delta_0 + \frac{\gamma A}{2} + \frac{\Delta_0}{2} - \frac{\Delta}{2}$

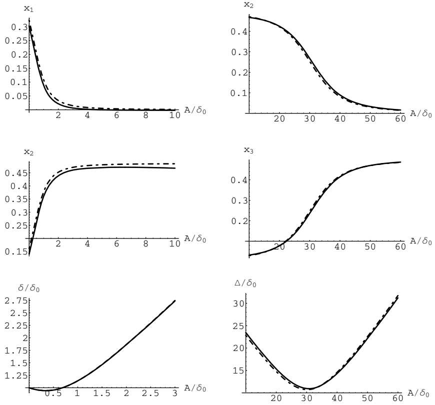


Figure: Approximated (dot-dashed) and numerical (solid) solutions, with $\epsilon = 0.17$, $\beta = 1/4$, and $\delta_0/\Delta_0 = 1/32$.

Some specific values

From approximated solutions:

- Solutions near lower resonance: $x_1 = x_{10}$, $x_2 = x_{20}$, $x_3 = x_{30}$, $\delta = \delta_0$, and $\Delta = \Delta_0$ if $A = 0$.
- Lower resonance condition: $A_l = \delta_0/(3 - 3\epsilon^2) \simeq 0.34\delta_0$
- $\delta_l = 2\sqrt{2}\delta_0/3 \simeq 0.94\delta_0$
- Higher resonance condition: $A_h = \Delta_0(1 - 2\epsilon^2) \simeq 0.94\Delta_0$
- $\Delta_h = \Delta_0\sqrt{1 - (1 - 2\epsilon^2)^2} \simeq 0.34\Delta_0$

Patterns of mixing

$$|U_{ij}|^2_v \sim \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}, |U_{ij}|^2_l \sim \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

$$|U_{ij}|^2_b \sim \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}, |U_{ij}|^2_h \sim \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix},$$

$$|U_{ij}|^2_d \sim \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

v=vacuum, l=lower resonance, b=between lower and higher resonance ($A \sim 10\delta_0$ here), h=higher resonance, and d=dense matter ($A \to \infty$).

Summary and Outlook

- An alternative parametrization is applied to neutrino mixing
- Evolution equations are derived
- Mixing parameters in matter are related to that in vacuum
- Intriguing patterns of the physical quantities can be investigated
- Apparent invariant relations
- Might be useful for a general study of neutrino mixing and RGE (renormalization Group Equation) running of mixing parameters.