$\mathcal{S}$-Matrix: More than meets the action

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Outline

- $\mathcal{S}$-Action $\leftrightarrow$ $\mathcal{S}$-Matrix ($\mathcal{S} = \mathcal{S}$)
- Hiddeny symmetries (Dual conformal) ($\mathcal{S} < \mathcal{S}$)
- Gravity =(Yang-Mills)^2 ($\mathcal{S}?\mathcal{S}$)

Feynman: $\mathcal{S}$ stands for action principle

Einstein-Hilbert (1915)

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R$$

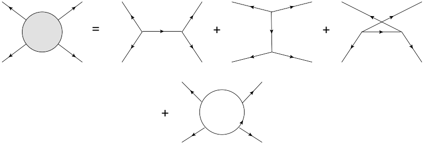
Yang-Mills (1954)

$$S = \frac{1}{g^2}\int d^4x\frac{1}{2}tr\{F_{\mu\nu}F^{\mu\nu}\}$$

Maximally super Yang-Mills (1977) Brink, Gliozzi, Scherk, Schrwaz, Olive

$$S = \frac{1}{g^2}\int d^4x tr\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}^{\dot{\alpha}}_A \not{D}_{\dot{\alpha}\beta}\lambda^{\beta A} - i\lambda^A_\alpha \not{D}^{\alpha\dot{\beta}}\bar{\lambda}_{A\dot{\beta}} + \frac{1}{2}(D_\mu\bar{\phi}_{AB})(D^\mu\phi^{AB})$$
$$-\sqrt{2}\bar{\phi}_{AB}\{\lambda^{\alpha A},\lambda^B_\alpha\} - \sqrt{2}\phi^{AB}\{\bar{\lambda}^{\dot{\alpha}}_A,\bar{\lambda}_{\dot{\alpha}B}\} + \frac{1}{8}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\}$$

- Perturbative definition of (Unitarity) $\mathcal{S}$-Matrix



- Non-Perturbative definition. (Sometimes = classical)
- Manifest symmetries: Global + Local
  (Physical) (Non-Physical)

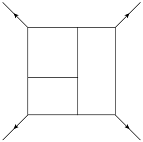
Ok so what has the action principle taught us about $\mathcal{S}$-matrix?

There are better things in life to do!

- Four-point Yang-Mills amplitude

$$
\begin{aligned}
A_{4s} \quad = \quad & -\frac{2}{s}\left(\begin{array}{l}
-\frac{s}{4}(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-\frac{u}{2}(\epsilon_1\cdot\epsilon_2)(\epsilon_4\cdot\epsilon_3)\\
+(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_3)(\epsilon_1\cdot\epsilon_3)+(\epsilon_1\cdot k_2)(\epsilon_3\cdot k_4)(\epsilon_2\cdot\epsilon_4)\\
+(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_4)(\epsilon_3\cdot\epsilon_4)+(\epsilon_4\cdot k_2)(\epsilon_3\cdot k_1)(\epsilon_1\cdot\epsilon_2)\\
-(\epsilon_1\cdot k_2)(\epsilon_4\cdot k_3)(\epsilon_2\cdot\epsilon_3)-(\epsilon_3\cdot k_4)(\epsilon_2\cdot k_1)(\epsilon_1\cdot\epsilon_4)\\
-(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_3)(\epsilon_3\cdot\epsilon_4)-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_2)
\end{array}\right)\\
& -\frac{2}{t}\left(\begin{array}{l}
-\frac{t}{4}(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-\frac{u}{2}(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\\
+(\epsilon_1\cdot k_4)(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_4)+(\epsilon_2\cdot k_3)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_3)\\
+(\epsilon_1\cdot k_3)(\epsilon_4\cdot k_2)(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_1)(\epsilon_1\cdot\epsilon_4)\\
-(\epsilon_1\cdot k_2)(\epsilon_4\cdot k_3)(\epsilon_2\cdot\epsilon_3)-(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_4)(\epsilon_1\cdot\epsilon_4)\\
-(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_3)(\epsilon_3\cdot\epsilon_4)-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_2)
\end{array}\right)
\end{aligned}
$$

- Three-point gravity amplitude: About 100 terms.

- $\mathcal{N}=8$ Sugra four-point three-loop  About 10^{20} terms.

Why?

Intermediate computations are off-shell$\rightarrow$

- 1. Off-shell objects are not gauge invariant. (The price you pay for beauty)
- 2. Ghosts: makes getting zero a tough job.
- 3. Unitarity is not manifest, only in the very end.

Which brings us to the ultimate question......

Why was gauge invariance a good idea in the first place ?

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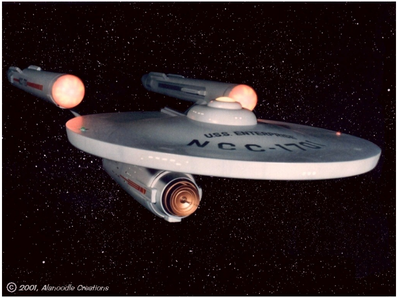
- It is ignorant of all irrelevant properties of the action, i.e. gauge invariance.
- It does not have to be derived from an action! (1st quantized, 2nd quantized, ... , guessing)

Space, the initial frontier!



- The space should form a representation of the symmetry of the theory.
  x for Poincare and x, θ for super Poincare
- It should covariantly parameterise all on-shell degrees of freedom.
  All momenta, polarization and particle species $\rightarrow$ Lorentz vectors are bad idea.
- The coordinates should all be independent, i.e. no self conjugacy.
  One cannot have $\{z^{\mathcal{M}}, z^{\mathcal{N}}] = \delta^{\mathcal{M}\mathcal{N}}$

4D Lorenz group $SO(1,3) \sim SL(2,C)$

$$P^\mu = P_{\alpha\dot{\alpha}}(\gamma^\mu)^{\alpha\dot{\alpha}}$$
$$P^2 = 0 \rightarrow det\,(P_{\alpha\dot{\alpha}}) = 0$$

$P_{\alpha\dot{\alpha}}$ is a rank 1 matrix:

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

Recall Four-point Yang-Mills amplitude

$$A_{4s} = -\frac{2}{s}\begin{pmatrix} -\frac{s}{4}(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-\frac{u}{2}(\epsilon_1\cdot\epsilon_2)(\epsilon_4\cdot\epsilon_3) \\ +(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_3)(\epsilon_1\cdot\epsilon_3)+(\epsilon_1\cdot k_2)(\epsilon_3\cdot k_4)(\epsilon_2\cdot\epsilon_4) \\ +(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_4)(\epsilon_3\cdot\epsilon_4)+(\epsilon_4\cdot k_2)(\epsilon_3\cdot k_1)(\epsilon_1\cdot\epsilon_2) \\ -(\epsilon_1\cdot k_2)(\epsilon_4\cdot k_3)(\epsilon_2\cdot\epsilon_3)-(\epsilon_3\cdot k_4)(\epsilon_2\cdot k_1)(\epsilon_1\cdot\epsilon_4) \\ -(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_3)(\epsilon_3\cdot\epsilon_4)-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_2) \end{pmatrix}$$

$$-\frac{2}{t}\begin{pmatrix} -\frac{t}{4}(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-\frac{u}{2}(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3) \\ +(\epsilon_1\cdot k_4)(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_4)+(\epsilon_2\cdot k_3)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_3) \\ +(\epsilon_1\cdot k_3)(\epsilon_4\cdot k_2)(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_1)(\epsilon_1\cdot\epsilon_4) \\ -(\epsilon_1\cdot k_2)(\epsilon_4\cdot k_3)(\epsilon_2\cdot\epsilon_3)-(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_4)(\epsilon_1\cdot\epsilon_4) \\ -(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_3)(\epsilon_3\cdot\epsilon_4)-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)(\epsilon_1\cdot\epsilon_2) \end{pmatrix}$$

Now

$$A(i^-,j^-,l^+,k^+)=\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

with $\langle ij\rangle=\lambda_i^{\alpha}\lambda_{j\alpha}$.

Twistor space defines conformally compactified Minkowski space Fundamentals of $SU(2,2|\mathcal{N})$

$$[z^{\mathcal{M}}, \bar{z}_{\mathcal{N}}\} = \delta^{\mathcal{M}}_{\mathcal{N}}$$

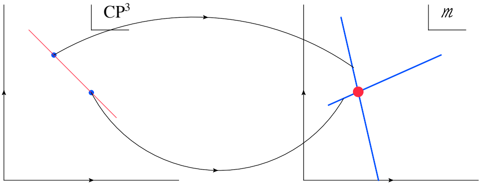
$$\left(\begin{array}{c} z^{\mu} \\ z^{m} \end{array}\right) = \left(\begin{array}{c} \lambda^{\alpha} \\ \mu^{\dot{\beta}} \\ \eta^{m} \end{array}\right)$$

Incident relation

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_{\alpha}$$

A point in twistor space defines a null line in Minkowski space

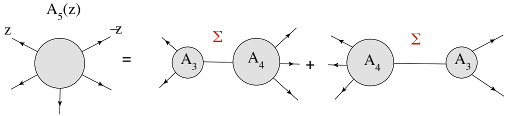
$$\begin{aligned} \mu^{\dot{\alpha}} &= x_1^{\alpha\dot{\alpha}}\lambda_{\alpha},\ \mu^{\dot{\alpha}} = x_2^{\alpha\dot{\alpha}}\lambda_{\alpha} \\ &\rightarrow (x_2 - x_1)^{\alpha\dot{\alpha}}\lambda_{\alpha} = 0 \\ &\rightarrow (x_2 - x_1)^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}} \end{aligned}$$



Local $\leftrightarrow$ Non-Local

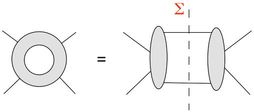
What about unitarity?

- Tree level: Amplitude factorizes.
  Recursion relations (Britto, Cachazo, Feng, Witten)
  The poles of a tree-level S-matrix must come from propagators going on shell



- Loop level1: Unitarity

$$S^{\dagger}S = (1+iT)^{\dagger}(1+iT) = 1 \rightarrow TT^{\dagger} = i(T^{\dagger} - T)$$

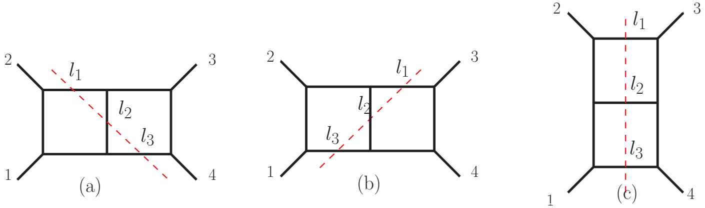


- Loop level2: Generalized Unitarity

Modern approach to loops: Generalized Unitarity (Bern, Dixon, Dunbar, Kosower)

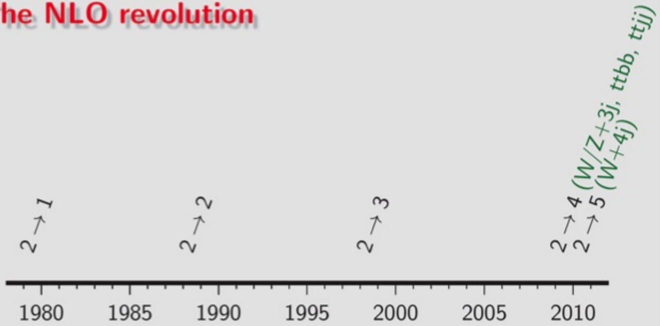
$$A_n^L|_{cuts} = \sum_i c_i I_i|_{cuts}$$

I_i are scalar integrals: Example $I_4 = \int dl^4 \frac{1}{l^2(l-k_1)^2(l-k_1-k_2)^2(l+k_4)^2}$
Coefficients c_i are fixed by matching all cuts.



Thus all information for loop $\mathcal{S}$-matrix elements are in tree.

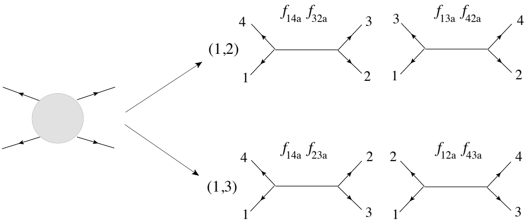
The NLO revolution



2010: NLO $W+4j$ [BlackHat: Berger et al, preliminary] [unitarity]

Symmetry+Unitarity→ Non-abelian gauge theory

- Three-point:
  For Poincaré: $A_3 \sim \delta^4(P)(\ \langle ij\rangle, [ij]\ )$.
  For dilatation+helicity: $A_3 \sim \delta^4(P)\left(\sum_{i\neq j}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right)$
  → A_3 is totally antisymmetric, i.e. a totally anti-symmetric "coupling constant "f^{abc} .
- The recursion relation has to be consistent



Implies Jacobi relation $f^{c[ab}f_c{}^{d]e} = 0 \rightarrow f^{abc} = tr(T^c[T^a, T^b])$.
It is based on a Lie-2 algebra.

All ingredients of the action is recovered.

Using global symmetries and unitarity$\rightarrow$ one gets the action $\mathcal{S}$.

- Four-dimensional (super)Yang-Mills, (super) Gravity. (Cachzao, Kaplan)
- Three-dimensional $\mathcal{N} = 8, 6$ Chern-Simons matter theory (Huang, Lipstien)

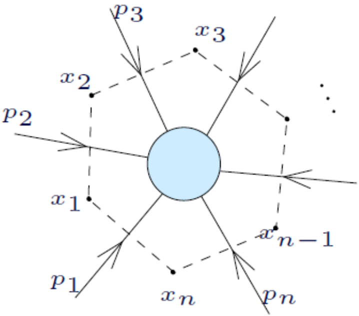
No scalar ϕ^n theories.

Hidden Symmetries.($\mathcal{S} > \mathcal{S}$)

At planar level $\mathcal{N} = 4$ Super Yang-Mills SU(N) with $N \to \infty$
Dual conformal symmetry: (Drummond, Henn, Korchemsky and Sokatchev 08)

$$x_i - x_{i+1} = p_i$$

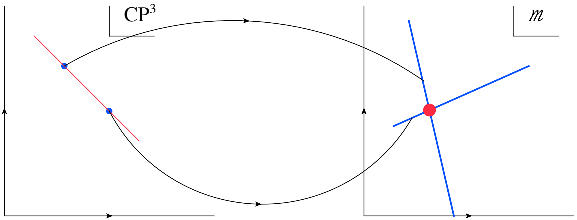
Momentum conservation is trivially satisfied $\sum_i^n p_i = o$



$$A_n(p_1, p_2, \cdots, p_n) \to \hat{A}_n(x_1, x_2, \cdots, x_n)$$

$$I\left[\hat{A}_n(x_1, x_2, \cdots, x_n)\right] = x_1^2 x_2^2 \cdot \cdot x_n^2 \hat{A}_n(x_1, x_2, \cdots, x_n)$$

Recall that in twistor space



Dual conformal symmetry must have a simpler expression using twistors.

Indeed, in twistor space ordinary (super)conformal + dual (super)conformal $\rightarrow$ infinitely many conserved charge.

- Yangian symmetry (Drummond, Henn, Plefka 09)

$$
\begin{aligned}
(superconformal)\ J^0 &= \sum_i z_i^{\mathcal{M}} \frac{\partial}{\partial z_i^{\mathcal{N}}} \\
(dualsuperconformal)\ J^1 &= \sum_{i<j} z_i^{\mathcal{M}} \frac{\partial}{\partial z_i^{\mathcal{C}}} z_j^{\mathcal{C}} \frac{\partial}{\partial z_j^{\mathcal{M}}} \\
&\quad \cdots\cdots
\end{aligned}
$$

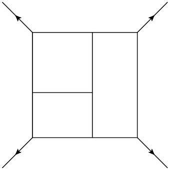
  This hints at the amplitude can be exactly determined.

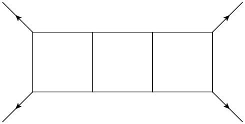
- The only Yangian invariant object

$$\langle i, j, k, l\rangle = \epsilon_{ABCD} z_i^A z_j^B z_k^C z_l^D$$

- All loop integrand of planar amplitudes determined (Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka 10)

Three-loop four-point integrand is fixed.





Some Caveats

- The loop amplitudes need regulator (infrared)
- From the string arguments, mass is the natural choice. (Alday, Henn, Plefka, Schuster 09)
- Six-dimension massless amplitude= four-dimensions massive amplitudes. (Dennen, Huang 10)

The simplest theory in four-dimensions

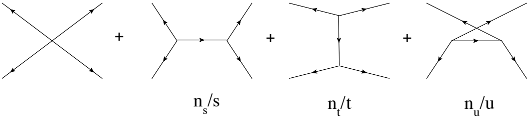
$$\mathcal{N}=4 \ \text{super} \ \text{Yang}-\text{Mills}$$

The simplest theory in three-dimensions (Huang, Lipstein 10)
(Gang, Huang, Koh, Lee, Lipstein 10)

$$\mathcal{N}=6 \ \text{Chern}-\text{Simons} \ \text{matter}$$

Gravity=$(\text{Yang-Mills})^2$

A new "duality" was found for Yang-Mills (Bern, Carrasco, Johansson 08)(BCJ)
Consider the four-point tree amplitude



Judicially pushing the contact term into others, ($1 = s/s = t/t = u/u$)
One can achieve

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

where $c_u = f^{42b} f^{b31}, \ c_s = f^{12b} f^{b34}, \ c_t = f^{41b} f^{b23}$

$$\begin{aligned} c_u &= c_s - c_t \\ n_u &= n_s - n_t \end{aligned}$$

A color kinematic duality

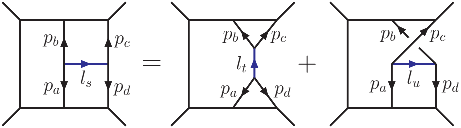
This holds for all tree amplitude.(Bern, Carrasco, Dennen, Huang, Kiermaier 10)
In general there is always a choice where

$$\begin{aligned} c_i &= c_j - c_k \\ n_i &= n_j - n_k \end{aligned}$$

Furthermore one gets gravity!

$$A_{YM} = \sum_i \frac{c_i n_i}{\prod_{i_\alpha} P_{i_\alpha}} \rightarrow A_{Gravity} = \sum_i \frac{n_i n_i}{\prod_{i_\alpha} P_{i_\alpha}}$$

BCJ representation exists at loops as well.

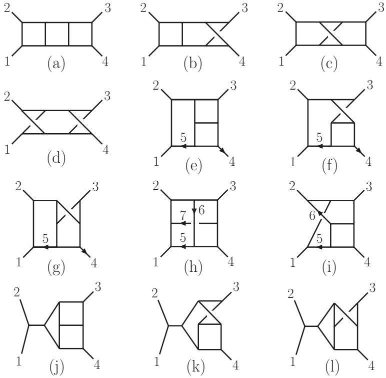


Partially explains the same UV divergence of $\mathcal{N} = 8$ Sugra and $\mathcal{N} = 4$ Super Yang-Mills

UV divergence

(Bern, Carrasco, Johansson 10)

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\mathcal{N}=8$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$[s\,(-\tau_{35}+\tau_{45}+t)-t\,(\tau_{25}+\tau_{45})+u\,(\tau_{25}+\tau_{35})-s^2]/3$
(h)	$[s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2]/3$
(i)	$[s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2]/3$
(j)-(l)	$s(t-u)/3$



What was the action doing all this time?

To give BCJ representation, the Lagrangian is to be modified

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 \cdots\cdots\cdots$$

However, the modification is ZERO

$$\begin{aligned}\mathcal{L}'_5 \;=\; & -\frac{1}{2}g^3(f^{a_1 a_2 b} f^{b a_3 c} \\ & \qquad + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c}) f^{c a_4 a_5} \\ & \qquad \times \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3 \mu} \frac{1}{\Box}(A^{a_4 \nu} A^{a_5 \rho}) \,.\end{aligned}$$

That is

$$\text{Yang} - \text{Mills} \; + 0 \rightarrow \text{Gravity}$$

Conclusion

- Action may not always be $\left\{ \begin{array}{c} \textit{Convenient} \\ \textit{Sufficient} \\ \textit{Available} \end{array} \right.$ for perturbative physics.
- It pays to be physical.
- Hidden symmetries for planar conformal theories.
- Hidden relationship between gravity and Yang-Mills.

In the near future

- Completely solve 3D $\mathcal{N} = 6$
- Before somebody completely solves 4D $\mathcal{N} = 4$
- Is there anything manifest dual conformal quantized system ?
- Is there anything manifest BCJ duality quantized system ?

In the back of our heads

If the action does not carry all the symmetries present in perturbative physics,
How solid are the statements based on symmetries of the action? (renormalizability)