will charged particles exhibit the Brownian motion in quantum vacuum

December 15, 2008

introduction

- imagine that a free charged particle couples to the vacuum state of the quantized electromagnetic field.
- the vacuum fluctuations of the EM field $\Rightarrow$ fluctuating motion of the charge.
- but will the velocity dispersion grows with time as in the typical Brownian motion?
- if so, where does the energy come from?
- if not, why?

Brownian motion

- the Brownian motion is described by the Langevin equation,

$$m\frac{dv}{dt} + m\gamma v = \xi(t)\,.$$

- γ phenomenologically describes dissipation due to interaction with the environment.
- $\xi(t)$ is some fluctuating force from the environment. It satisfies some statistical properties $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = C\,\delta(t - t')$ for stationary white-noise background.
- $\langle \cdots \rangle$: averaging procedure relevant to the problem.

Brownian motion

- the velocity dispersion is

$$\langle \Delta v^2(t) \rangle = \frac{1}{2\beta} \left[1 - e^{-2\gamma t} \right] .$$

$\because C = 2m\gamma/\beta$ due to equal partition theorem.

- at early time $t \ll \gamma^{-1}$, before dissipation sets in, particle's motion mainly results from environmental noise.
- $\langle \Delta v^2(t) \rangle = \frac{C}{m^2} t$ increases with time.
- at late time, dissipation becomes dominant till the energy flow is balanced.
- $\langle \Delta v^2(\infty) \rangle = \dfrac{1}{m\beta}$.

Brownian motion

- so without dissipation, $\langle \Delta v^2(t) \rangle$ will grow indefinitely, or
- at zero temperature $\beta \to \infty$, $\langle \Delta v^2(t) \rangle$ vanishes? make sense in the sense of classical physics, $\because$ the classical thermal fluctuations of the environment vanish...
- what about the quantum-field environment?

how does a charged particle move in the quantized EM field?

- assume that a quantum-mechanical charged particle interacts with the electromagnetic field and moves nonrelativistically.
- what to expect?
- even the intrinsic quantum fluctuations of the particle is ignored, the EM vacuum should still drive it into zigzag motion.

Lagrangian

the Lagrangian in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, is

$$L[\mathbf{q}, \mathbf{A}_{\mathrm{T}}] = \frac{1}{2} m \dot{\mathbf{q}}^2 - V(\mathbf{q}) - \frac{1}{2} \int d^3\mathbf{x}\, d^3\mathbf{y}\ \varrho(x; \mathbf{q}) G_c(\mathbf{x}, \mathbf{y}) \varrho(y; \mathbf{q}) + \int d^3\mathbf{x} \left[\frac{1}{2} (\partial_\mu \mathbf{A}_{\mathrm{T}})^2 + \mathbf{j} \cdot \mathbf{A}_{\mathrm{T}} \right],$$

with

$$\nabla^2 G_c(\mathbf{x}, \mathbf{y}) = -\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \varrho(\mathbf{x}; \mathbf{q}(t)) = e\, \delta^{(3)}(\mathbf{x} - \mathbf{q}(t)),$$
$$\mathbf{j}(\mathbf{x}; \mathbf{q}(t)) = e\, \dot{\mathbf{q}}(t)\, \delta^{(3)}(\mathbf{x} - \mathbf{q}(t)).$$

formalism 1/5

- in the open-system approach, the dynamics of the charged particle is formally described by its reduced density operator

$$\begin{aligned}\rho_r(\mathbf{q}_f, \mathbf{q}'_f, t_f) &= \mathrm{Tr}_{\mathbf{A}_\mathrm{T}}\left\{\rho_{tot}(\mathbf{q}, \mathbf{A}_\mathrm{T}, t_f)\right\} \\ &= \mathrm{Tr}_{\mathbf{A}_\mathrm{T}}\left\{U(t_f, t_i)\rho_{tot}(\mathbf{q}, \mathbf{A}_\mathrm{T}, t_i)U^{-1}(t_f, t_i)\right\}\end{aligned}$$

where $U(t_f, t_i)$ is the unitary evolution operator of the total system from t_i to t_f.

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formalism 2/5

- in terms of path integral, the evolution of the initial state, together with the "trace" can be written as

$$\rho_r(\mathbf{q}_f, \mathbf{q}'_f, t_f) = \int d\mathbf{q}_i d\mathbf{q}'_i \int_{\mathbf{q}_i}^{\mathbf{q}_f} \mathcal{D}\mathbf{q}^+ \int_{\mathbf{q}'_i}^{\mathbf{q}'_f} \mathcal{D}\mathbf{q}^- \times \exp\left[i\left(S_0[\mathbf{q}^+] - S_0[\mathbf{q}^-] + S_{inf}[\mathbf{q}^+, \mathbf{q}^-]\right)\right] \times \rho_e(\mathbf{q}_i, \mathbf{q}'_i, t_i)$$

where $\rho_e(\mathbf{q}_i, \mathbf{q}'_i, t_i)$ is the initial state of the particle.

formalism 3/5

- all influence from the quantized EM field is contained in $S_{inf}[\mathbf{q}^+, \mathbf{q}^-]$ in terms of G_R^{ij} and G_H^{ij} of the EM field,

$$S_{inf}[\mathbf{q}^+, \mathbf{q}^-] = -\frac{1}{2}\int_{t_i}^{t_f} ds \int_{t_i}^{s} ds' \, (j^+ - j^-) G_R(s, s')(j'^+ + j'^-)$$
$$-\frac{i}{2}\int_{t_i}^{t_f} ds \int_{t_i}^{s} ds' \, (j^+ - j^-) G_H(s, s')(j'^+ - j'^-)$$

with $j = j(s)$ and $j' = j(s')$, and

$$G_R^{ij}(x, x') = i\,\theta(t - t')\langle [A_{\mathrm{T}}^i(x), A_{\mathrm{T}}^j(x')]\rangle \,,$$
$$G_H^{ij}(x, x') = \frac{1}{2}\langle \{A_{\mathrm{T}}^i(x), A_{\mathrm{T}}^j(x')\}\rangle \,.$$

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formalism 4/5

- $e^{iS_{inf}}$ is called influence functional.
- the imaginary part of $S_{inf}[\mathbf{q}^+, \mathbf{q}^-]$ can be formally expressed by

$$e^{i\,\mathrm{Im}\{S_{inf}\}} = \int \mathcal{D}\xi \; \mathcal{P}[\xi] e^{i\xi(j^+ - j^-)} \,.$$

- ξ is some stochastic variable, satisfying

$$\langle \xi^i \rangle = 0 \,, \qquad \langle \xi^i(t) \xi^j(t') \rangle = G_H^{ij}(\mathbf{x}, t; \mathbf{x}, t') \,.$$

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- define the stochastic effective action S_{seff}

$$S_{seff}[\mathbf{q}^+,\mathbf{q}^-] = S_0[\mathbf{q}^+] - S_0[\mathbf{q}^-] + \\ + \mathrm{Re}\left\{S_{inf}[\mathbf{q}^+,\mathbf{q}^-]\right\} - \xi(j^+ - j^-)\,.$$

- taking variation with respect to $\Delta = \mathbf{q}^+ - \mathbf{q}^-$

$$\left.\frac{S_{seff}}{\delta\Delta}\right|_{\Delta=0} = 0$$

in the limit $\Delta \to 0$ yields the stochastic equation of motion of a charged particle influenced by the quantized EM field.

equation of motion 1/4

the equation of motion takes the form

$$
\begin{aligned}
& m\ddot{q}^i + \nabla^i V(\mathbf{q}(t)) + e^2 \nabla^i G_c(\mathbf{q}(t), \mathbf{q}(t)) \\
& \qquad + e^2 \left[\delta^{ij} \frac{d}{dt} - \dot{q}^j(t) \nabla^i \right] \int dt' \; G_R^{jk} \left(\mathbf{q}(t), t; \mathbf{q}(t'), t' \right) \dot{q}^k(t') \\
& = -e^2 \left[\delta^{ij} \frac{d}{dt} - \dot{q}^j(t) \nabla^i \right] \xi^j(t) \,,
\end{aligned}
$$

and since the intrinsic quantum fluctuation of the particle has been ignored, it only describes the external influence on particle's motion.

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equation of motion 2/4

observe that

$$A^i_{LW} = e \int dt' \; G^{ij}_R \left(\mathbf{q}(t), t; \mathbf{q}(t'), t' \right) \dot{q}^j(t') \, ,$$

is the Lienard-Wiechert potential in the Coulomb gauge.
It has two parts,

- local backreaction ⇔ self-force,
- non-local backreaction ⇔ non-Markovian/retarded effect if some boundary is present (not our concern in the present case).

equation of motion 3/4

the EoM can be written as

$$m\ddot{q}^i - m\tau_e \dddot{q}^i + \nabla^i V(\mathbf{q}) = f_s^i\,, \quad \text{with}$$

$$f_s^i = e\left[e_{\rm T}^i + \epsilon^{ijk} q^j(t) b^k\right] \leftrightarrow \text{stochastic transverse Lorentz force.}$$

- $e_{\rm T}^i$ and b^i: stochastic transverse EM fields $\leftarrow$ "stochastic" vector potential $\xi^i \Leftrightarrow$ vacuum fluctuations of the EM field.

equation of motion 4/4

- the full dynamics involves
  - ordinary force, $-\nabla^i V$,
  - self-force, third-order time derivative of position, $m\tau_e \dddot{q}^i$,
  - non-Markovian force, $\mathcal{F}_R^i$,
  - stochastic force, f_s^i.
- the EoM describes stochastic, fluctuating motion.
- then how does the velocity fluctuations behaves?

velocity fluctuations

let $q^i = \bar{q}^i + z$, where $\bar{q}^i = \langle q^i \rangle$ and z is fluctuations away from the mean trajectory. then EoM can be split into two parts,

$$m\ddot{\bar{q}}^i - m\tau_e \dddot{\bar{q}}^i + k\bar{q} = 0\,,$$
$$m\ddot{z}^i - m\tau_e \dddot{z}^i + kz = f_s^i\,,$$

if we choose $V(\mathbf{q})$, say, to a harmonic potential.

- the 1st equation describe the "classical" evolution of the charge.
- the 2nd equation describe the dynamics of charge's position fluctuations. (nonclassical)

velocity fluctuations

if we ignore the dissipation term for the moment, the velocity fluctuations in the i direction are given by

$$\langle \Delta v_i^2(t) \rangle = \frac{e^2}{m^2} \int_0^t dt' dt'' \, \frac{1}{2} \langle \{ e_{\mathrm{T}}^i(t'), e_{\mathrm{T}}^i(t'') \} \rangle$$
$$= \frac{e^2}{m^2} \int_0^t dt' dt'' \, \frac{\partial^2}{\partial t' \partial t''} \left[G_H^{ii}(t' - t'') \right],$$

in the nonrelativistic limit.

- the integrals are not well defined b/c $G_H^{ii}(t' - t'')$ is singular when $t' = t''$.

velocity fluctuations

- conceptually the divergence results from the fact that all high frequency modes are included.
- in another word, we are probing infinitesimal scales.
- introduce the smeared correlation function

$$\widetilde{G}(t-t') = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \; f(t,s) f(t',s') G(s-s') \, ,$$

- the smearing function $f(s,t)$ is required to have the properties
  1. $\int_{-\infty}^{\infty} ds \; f(t,s) = 1,$
  2. $f(t,s) \geq 0,$
  3. $f(t,s)$ is smooth.

velocity fluctuations

- an example:

$$f(t,s) = \frac{\sigma}{\pi} \frac{1}{(t-s)^2 + \sigma^2} ,$$

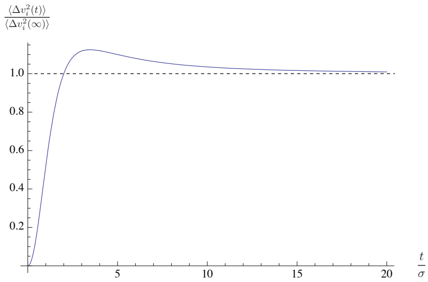
  with σ characterizing the scale of smearing.
- operationally/phenomenologically it describes the profile of the particle/probe, interaction switching, and so on.
- that should be more or less what we see for the correlation with a probe whose scale is about $\mathcal{O}(\sigma)$.
- then

$$\ddot{\widetilde{G}}{}_H^{ii}(\tau) = -\frac{1}{\pi^2} \frac{\tau^4 - 24\sigma^2\tau^2 + 16\sigma^4}{(\tau^2 + 4\sigma^2)^4} .$$

velocity fluctuations

- so the velocity dispersion becomes

$$\langle \Delta v_i^2(t) \rangle = \frac{e^2}{m^2} \frac{1}{12\pi^2} \frac{t^2(t^2 + 12\sigma^2)}{\sigma^2(t^2 + 4\sigma^2)^4}.$$



velocity fluctuations

- even without dissipation, the velocity dispersion saturates,

$$\langle \Delta v_i^2(\infty) \rangle = \frac{e^2}{m^2} \frac{1}{12\pi^2} \frac{1}{\sigma^2} = \frac{1}{12\pi^2} \frac{r_e}{\sigma} \frac{1}{m\sigma}$$
$$= \mathcal{O}\left(\frac{r_e}{\sigma} \langle \Delta v_i \rangle_{QM}\right).$$

- the saturated value is small, so the dissipation owing to velocity dispersion is never significant.
- justify the assumption that the dissipation can be ignored.
- then what makes it different from the Brownian particle?
- first note that the stochastic noise is colored. what is the implication?

velocity fluctuations

- next note that the work done by noise to the charged particle seems to balance itself in the end. why?
- let us check the energy flow. the power done by the noise is

$$\langle P_s(t)\rangle = 3\frac{e^2}{m}\int_0^t ds\, \frac{\partial^2}{\partial t \partial s}\left[\widetilde{G}_H^{ii}(t-s)\right]$$
$$= -\frac{e^2}{m}\frac{1}{\pi^2}\frac{t(t^2-12\sigma^2)}{(t^2+4\sigma^2)^3}\,.$$

- at late time, $\langle P_s(\infty)\rangle = 0$. it seems reasonable, consistent with energy balancing...

velocity fluctuations

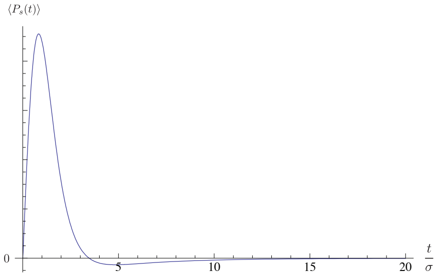


Figure: $\langle P_s(t)\rangle$ vs t.

- from time variation of the mean power, we see that it acquires positive values and then becomes negative after $t \sim \mathcal{O}(\sigma)$

velocity fluctuations

- the external field/noise pumps positive energy into the particle in the beginning, but it can extract energy out of the particle as well.
- that is quite different from the case of the Brownian particle.
- so why and how does the noise pull the energy out?
- let us check the correlation of the stochastic noise...

velocity fluctuations

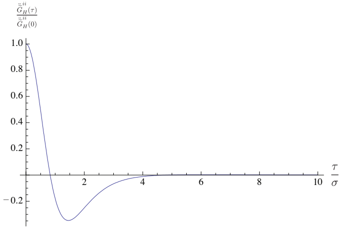


Figure: $\frac{\partial^2}{\partial t \partial s} \widetilde{G}_H^{ii}(t-s)$ vs τ with $\tau = t - s$.

- observe that the time variation of noise implies that the spectrum of noise is colored.
- there exist regions where correlation is negative.

velocity fluctuations

- it seems to imply the (negative) correlation gradually pulls particle's motion out of phase with the noise.
- so the power gradually becomes negative.
- in turn, it prevents the velocity fluctuations from growing forever!
- compared with the Brownian particle, the spectrum of noise is white so the correlation function is a delta function.
- so the driving force is always in phase with motion.
- the input power is always positive and independent of time.
- that causes $\langle \Delta v_i^2(t) \rangle$ to increase forever.

a quick conclusion

- therefore the unusual property of vacuum fluctuations of the EM field seem to contribute to
  - colored noise $\Longleftrightarrow$ non-local correlation, and
  - existence negative correlation.
- but is it the whole story?
- one lesson we learned is that dissipation can be reasonably ignored even we look into long-time dynamics of the fluctuations of charge's motion.

finite temperature

- at finite temperature β^{-1}, the smeared correlation function is

$$
\begin{aligned}
&\partial_{tt'}\widetilde{D}^{ii}_{\sigma}(t-t') \\
&= \frac{\mathcal{A}}{2}\int_0^{\infty} d\omega\, \frac{\omega^3}{2}\, e^{-2\omega\sigma} e^{-i\omega\tau}\, \frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}} + \text{c.c.} \\
&= \frac{\mathcal{A}}{2}\left[-\frac{3}{(\tau-2i\,\sigma)^4} + \frac{1}{\beta^4}\,\psi^{(3)}\!\left(\frac{2\sigma+i\,\tau}{\beta}\right)\right] + \text{c.c.}\,,
\end{aligned}
$$

where $\mathcal{A}$ is angular contribution, $1/3\pi^2$.

- the vacuum contribution is included.

finite temperature

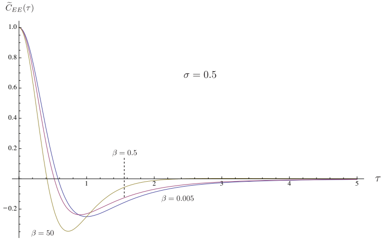


Figure: Time variation of the normalized smeared correlation function.

- it does not vary much over a wide range of temperature.
- negative correlation always exists for $\tau > \mathcal{O}(\sigma)$.

finite temperature

- even the finite temperature correction of correlation can always be negative.
- so we may draw similar conclusions as in the case of zero temperature.

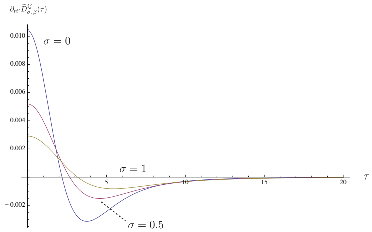


Figure: The finite temperature contribution with $\beta = 5$.

finite temperature

- in the high temperature limit, the velocity fluctuations reaches a constant

$$\langle \Delta v_i^2(\infty) = \frac{1}{6\pi^2}\frac{e^2}{m\sigma}\frac{1}{\beta} = \mathcal{O}(\frac{r_c}{\sigma}\frac{1}{\beta}) = \mathcal{O}(\frac{r_c}{\sigma}\langle \bar{v}_{th}^2\rangle)\,.$$

- so what makes the Brownian particle so special?
- in general, a Brownian particle is equivalent to the high-temperature, point-particle limit of the charged particle in a fictitious coupling $\lambda^2 e\, q_\mu A^\mu$ with the EM field where λ is some positive constant to make the dimension right.

finite temperature

- in this sense, the relevant correlation function is the smeared anticommutator of the vector potentials,

$$\widetilde{D}^{ii}_{\sigma}(\tau) = \frac{\mathcal{A}}{2}\int_0^\infty d\omega\,\frac{\omega}{2}\,e^{-2\omega\sigma}e^{-i\omega\tau}\,\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}} + \text{c.c.}$$
$$= \frac{\mathcal{A}}{2}\left[\frac{1}{2(\tau-2i\,\sigma)^2}+\frac{1}{\beta^2}\,\psi^{(1)}\left(\frac{2\sigma+i\,\tau}{\beta}\right)\right] + \text{c.c.}\,.$$

- in the high-temperature, point-particle limit, it reduces to

$$D^{ii}(\tau) = \frac{\pi}{\beta}\,\mathcal{A}\,\delta(\tau)\,.$$

- the key is that $\widetilde{D}^{ii}_{\sigma}(\tau)$ does not always permit negative correlation.

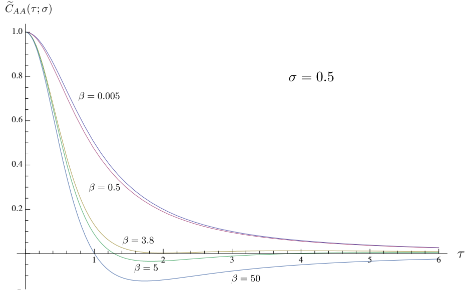


Figure: The time correlation of the vector potential.

- for sufficiently high temperature, the correlation function becomes positive definite.

- in the end the mean power does not always vanish,

$$\langle P_s(\infty)\rangle = \frac{\pi}{\beta},$$

- except in the zero temperature limit $\beta \to \infty$.
- in the high temperature limit, the velocity fluctuations grow linearly in time

$$\langle \Delta v_i^2(t) \propto \frac{\pi}{\beta}\, t\,.$$

- dissipation has to be taken into consideration in order to reach relaxation.
- comparing these two cases, we see the major difference lies in $\langle P_s(\infty)\rangle = 0$ or not.

- observe that

$$\langle P_s(\infty)\rangle \propto \int_{-\infty}^{\infty} d\tau\ \widetilde{C}^{ii}(\tau) = \lim_{k\to 0} k^n \rho^{ii}(k)\,.$$

here $\rho^{ij}(k)$ is the spectral density of the vector potential, defined by

$$\frac{1}{2}\langle\{A^i(t), A^j(t')\}\rangle = \int_0^{\infty} dk\ \rho^{ij}(k)\cos k(t-t')\,,$$

and

$$\rho^{ii}(k) = \frac{\mathcal{A}}{4}\, k\, \frac{1+e^{-\beta k}}{1-e^{-\beta k}}\,.$$

- in the $3+1$ dimensional spacetime, $n=2$ for the charged particle; $n=0$ for the Brownian particle.

finite temperature

- it is clear to see that

$$\lim_{k\to 0} k^n \rho^{ii}(k) = \begin{cases} 0\,, & \text{charged particle}\,, \\ \frac{\mathcal{A}}{2\beta}\,, & \text{Brownian particle}\,. \end{cases}$$

- so it depends on
  - dimension of spacetime,
  - coupling with the environment,
  - behavior of the spectral density of the environment in the zero energy limit.

conclusions

- the vacuum fluctuations of the EM field do drive the charged particle into fluctuating motion.
- the corresponding velocity dispersion of the particle does not increase indefinitely as the Brownian particle.
- the difference is related to whether the environment allows for negative correlation.
- the colored noise from the environment does not only pump energy to the particle, but also extracts energy out.
- if in the end the mean power done by the environment vanishes, particle's velocity dispersion may saturate with dissipation being safely ignored.
- all depends on if $\lim_{k\to 0} k^n \rho^{ii}(k)$ vanishes or not.