# On the Geometric Implications of a Flux Integral Representation for the Twin Paradox

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# Main Motivation:

Try to find an analogous expression

to the Aharonov-Bohm effect in quantum physics

of the twin effect in special relativity.

## Theorem I:

The length difference between two curves, $C_1$ and $C_0$, sharing identical end-points $\partial C_1 = \partial C_0 = \{A, B\}$, is equal to the integral of the geodesic curvatures $\kappa_g(C_\lambda)$ along all interpolating curves $C_\lambda$ (with common end-points $\partial C_\lambda = \{A, B\}$) with respect to any deformation surface $\Sigma$ bounded by $C_1$ and $C_0$.

$$\iint_\Sigma \kappa_g(C_\lambda)\ d^2a = \int_{C_1} dl - \int_{C_0} dl, \quad \partial\Sigma = C_1 \cup (-C_0). \tag{1}$$

## Theorem II:

The area difference between two surfaces, $S_1$ and $S_0$, sharing identical boundary contour $\partial S_1 = \partial S_0 = C$, is equal to the integral of twice of the mean curvatures $H(S_\lambda)$ over all interpolating surfaces $S_\lambda$ (with common boundary contour $\partial S_\lambda = C$) within the interior region, $\Omega$, of the closed surface formed by the union of $S_1$ and $-S_0$.

$$\iiint_\Omega 2H(S_\lambda)\ d^3\tau = \iint_{S_1} d^2a - \iint_{S_0} d^2a. \ \ \partial\Omega := S_1 \cup (-S_0). \tag{2}$$

- The starting point of our construction is to express the length difference between two curves sharing common end-points as a contour integral,

$$\begin{aligned} \Delta l &= l_1 - l_0 = \oint_{C=C_1\cup(-C_0)} dl \\ &= \oint_{C=C_1\cup(-C_0)} \sqrt{\left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2} d\sigma. \end{aligned} \tag{3}$$

- Here we take general parametrization of the curves $C_1$, $C_0$ as

$$\begin{aligned} C_1 \Rightarrow \ \vec{r}_1(\sigma) &:= (x_1(\sigma), y_1(\sigma), z_1(\sigma)), \\ C_0 \Rightarrow \ \vec{r}_0(\sigma) &:= (x_0(\sigma), y_0(\sigma), z_0(\sigma)). \end{aligned} \tag{4}$$

$$\begin{aligned} dl &= \frac{d\vec{r}}{dl} \cdot d\vec{r} =: \vec{A} \cdot d\vec{r}, \quad \oint_C dl = \oint_C \vec{A} \cdot d\vec{r}. \\ \Rightarrow \vec{A} &= \frac{d\vec{r}}{dl} = \frac{d\vec{r}/d\sigma}{dl/d\sigma} = \frac{\vec{r}'}{\sqrt{\vec{r}' \cdot \vec{r}'}}, \\ \Rightarrow \vec{A}(\vec{r}(\sigma)) &= \vec{A}(x(\sigma), y(\sigma), z(\sigma)). \end{aligned} \tag{5}$$

- We introduce a "fiberization" of the 3-dimensional space as follows:

$$\begin{aligned} x &= f(\xi,\eta,\sigma) := x_0(\sigma) + \xi\,[\Delta x(\sigma)] \\ y &= g(\xi,\eta,\sigma) := y_0(\sigma) + \eta\,[\Delta y(\sigma)] \\ z &= h(\xi,\eta,\sigma) := z_0(\sigma) + \left(\frac{\xi+\eta}{2}\right)[\Delta z(\sigma)]\,. \end{aligned} \tag{6}$$

- This "fiberization" can be viewed as introducing a two-parameter family of interpolating curves, $\vec{r}_{\xi\eta}(\sigma) := (f(\xi,\eta,\sigma), g(\xi,\eta,\sigma), h(\xi,\eta,\sigma))$,

$$\begin{aligned} \vec{r}_{00}(\sigma) &= (f(0,0,\sigma), g(0,0,\sigma), h(0,0,\sigma)) = \vec{r}_0(\sigma), \\ \vec{r}_{11}(\sigma) &= (f(1,1,\sigma), g(1,1,\sigma), h(1,1,\sigma)) = \vec{r}_1(\sigma). \end{aligned} \tag{7}$$

- If we treat the fiberization of the three-dimensional space as a change of coordinates, $(x,y,z) \Leftrightarrow (\xi,\eta,\sigma)$, then it is possible to extend the definition of the vector potentials as follows,

$$\vec{A}(x,y,z) := \frac{\vec{r}\,'_{\xi\eta}(\sigma)}{\sqrt{\left[\vec{r}\,'_{\xi\eta}(\sigma)\cdot\vec{r}\,'_{\xi\eta}(\sigma)\right]}}, \quad \vec{r}\,'_{\xi\eta}(\sigma) := \frac{\partial}{\partial\sigma}\vec{r}_{\xi\eta}(\sigma). \tag{8}$$

- For later convenience, it is useful to notice the following identity,

$$\frac{\partial}{\partial\lambda}\vec{r}_{\lambda\lambda} = \frac{\partial}{\partial\xi}\vec{r}_{\xi\eta}\bigg|_{\eta,\sigma} + \frac{\partial}{\partial\eta}\vec{r}_{\xi\eta}\bigg|_{\xi,\sigma} = \Delta\vec{r}. \tag{9}$$

- We choose a deformation surface, $\Sigma$, as a diagonal slice of the fibrization,

$$\begin{aligned} \Sigma \Rightarrow \vec{R}(\lambda,\sigma) &:= \vec{r}_{\lambda\lambda}(\sigma) = \left(f(\lambda,\lambda,\sigma), g(\lambda,\lambda,\sigma), h(\lambda,\lambda,\sigma)\right), \\ \partial\Sigma &= C_1 \cup (-C_0). \end{aligned} \tag{10}$$

- We can now use the Stokes' theorem to derive our main result,

$$\oint_C dl = \oint_C \vec{A}.d\vec{r} = \iint_\Sigma \left(\vec{\nabla}\times\vec{A}\right)\cdot d^2\vec{a} = \iint_\Sigma \left(\vec{\nabla}\times\vec{A}\cdot\vec{U}\right) d\lambda d\sigma. \tag{11}$$

- Here $\vec{U}$ stands for the normal vector of the deformation surface $\Sigma$,

$$\begin{aligned} \vec{U} &:= \frac{\partial\vec{R}}{\partial\lambda}\times\frac{\partial\vec{R}}{\partial\sigma} = \frac{\partial\vec{r}_{\lambda\lambda}}{\partial\lambda}\times\frac{\partial\vec{r}_{\lambda\lambda}}{\partial\sigma}. \qquad (12) \\ \left(\vec{\nabla}\times\vec{A}\right)\cdot\vec{U} &= \varepsilon_{lmn}(\partial_m A_n)\varepsilon_{ljk}(\partial_\lambda r_j)(\partial_\sigma r_k) \\ &= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})(\partial_m A_n)(\partial_\lambda r_j)(\partial_\sigma r_k) \\ &= (\partial_\lambda A_n)(\partial_\sigma r_n) - (\partial_\sigma A_n)(\partial_\lambda r_n) = (\partial_\lambda\vec{A})\cdot\vec{r}' - (\partial_\sigma\vec{A})(\Delta\vec{r}). \end{aligned}$$

- Recall that $\vec{A}$ is defined as the normalized tangent vector of the interpolating curve,

$$\partial_\lambda \vec{A} = \frac{(\Delta \vec{r}')(\vec{r}' \cdot \vec{r}') - \vec{r}'(\vec{r}' \cdot \Delta \vec{r}')}{(\vec{r}' \cdot \vec{r}')^{\frac{3}{2}}}, \quad (\partial_\lambda \vec{A}) \cdot \vec{r}' = 0. \tag{13}$$

$$\partial_\sigma \vec{A} = \frac{\vec{r}''(\vec{r}' \cdot \vec{r}') - \vec{r}'(\vec{r}' \cdot \vec{r}'')}{(\vec{r}' \cdot \vec{r}')^{\frac{3}{2}}}. \quad \vec{r}' := \frac{\partial \vec{r}_{\lambda\lambda}}{\partial \sigma}, \quad \vec{r}'' := \frac{\partial^2 \vec{r}_{\lambda\lambda}}{\partial^2 \sigma}. \tag{14}$$

- Consequently, we can verify that the integrand in Eq.(), is indeed given by the geodesic curvature of the interpolating curve $\vec{r}_{\lambda\lambda}$ with respect to the deformation surface $\Sigma_\lambda$,

$$(\vec{\nabla} \times \vec{A}) \cdot \vec{U} = \frac{(\Delta \vec{r} \cdot \vec{v})(\vec{v} \cdot \vec{a}) - (\Delta \vec{r} \cdot \vec{a})(\vec{v} \cdot \vec{v})}{|\vec{v}|^{\frac{3}{2}}}$$

$$= \frac{(\vec{v} \times \vec{a}) \cdot (\Delta \vec{r} \times \vec{v})}{|\vec{v}|^{\frac{3}{2}}} = \frac{(\vec{v} \times \vec{a}) \cdot \vec{U}}{|\vec{v}|^{\frac{3}{2}}} = \kappa_g(\vec{r}_{\lambda\lambda}). \tag{15}$$

Here we use the kinematic symbols, velocity $\vec{v} := \vec{r}'$, and acceleration $\vec{a} := \vec{r}''$.

- The proof of the Theorem II for the area difference follows similar idea of that of the length difference. We first rewrite the area difference as an integral over closed surface,

$$\Delta A = A_1 - A_0 = \iint_{\partial\Omega} \pm\, d^2 a = \iint_{\Sigma} \pm \left| \vec{R}_u \times \vec{R}_v \right| du\ dv. \tag{16}$$

- By taking general parametrizations of the surfaces $S_1$, $S_0$ as

$$\begin{aligned} S_1 &: \vec{R}_1(u,v) := (x_1(u,v), y_1(u,v), z_1(u,v)), \\ S_0 &: \vec{R}_0(u,v) := (x_0(u,v), y_0(u,v), z_0(u,v)), \end{aligned} \tag{17}$$

- we compute the surface area element from the cross-product of two tangent vectors,

$$\begin{aligned} d^2 a &= \left| \partial_u \vec{R} \times \partial_v \vec{R} \right| du\ dv \\ &= \sqrt{ \begin{vmatrix} \partial_u y & \partial_u z \\ \partial_u y & \partial_u z \end{vmatrix}^2 + \begin{vmatrix} \partial_u z & \partial_u x \\ \partial_u z & \partial_u x \end{vmatrix}^2 + \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_u x & \partial_u y \end{vmatrix}^2 }\ du\ dv. \end{aligned} \tag{18}$$

- If we wish to identify the area element as a flux density of a vector field, $\vec{E}$,

$$d^2a = \vec{E} \cdot d^2\vec{a} = E_1 dy \wedge dz + E_2 dz \wedge dx + E_3 dx \wedge dy, \tag{19}$$

then it is natural to define $\vec{E}$ as the unit normal vector of the surface

$$\vec{E} := \frac{\partial_u \vec{R} \times \partial_v \vec{R}}{\left| \partial_u \vec{R} \times \partial_v \vec{R} \right|}. \tag{20}$$

- To extend the definition of the vector field $\vec{E}$ throughout the 3-dimensional space, we in introduce a foliation of the 3-dimensional space as follow :

$$\begin{aligned} x &= F(u,v,\lambda) := x_0(u,v) + \lambda \Delta x(u,v), \\ y &= G(u,v,\lambda) := y_0(u,v) + \lambda \Delta y(u,v), \\ z &= H(u,v,\lambda) := z_0(u,v) + \lambda \Delta z(u,v). \end{aligned} \tag{21}$$

- Again, this foliation can be either thought as a family of interpolating surfaces,

$$S_\lambda \Rightarrow \vec{R}_\lambda(u,v) = \vec{R}_0(u,v) + \lambda \Delta \vec{R}(u,v), \quad \Delta \vec{R}(u,v) := \vec{R}_1(u,v) - \vec{R}_0(u,v), \tag{22}$$

or treated as a change of coordinates $(x,y,z) \Leftrightarrow (u,v,\lambda)$. In either case, we can apply the divergence theorem to Eq.(27), and obtain

$$\Delta A = \iint_{\partial\Omega} \vec{E} \cdot d^2\vec{a} = \iiint_{\Omega} \vec{\nabla} \cdot \vec{E} d^3\tau. \tag{23}$$

- It is know that for a given surface, the divergence of the unit normal vector is equal to double of the mean curvature

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{\partial_u \vec{R} \times \partial_v \vec{R}}{\left| \partial_u \vec{R} \times \partial_v \vec{R} \right|} = 2H(S_\lambda). \tag{24}$$

Thus this completes the proof of our second main result, Theorem II.

$$\Delta A = A_1 - A_0 = \iint_{\partial\Omega} d^2a = \iiint_{\Omega} 2H(S_\lambda)\ d^3\tau. \tag{25}$$