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Introduction to Quantum Finance

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Introduction to Quantum Finance

1. European call option pricing with stochastic calculus
2. European call option pricing with quantum mechanics and path integral
3. Quatum formalism for an isolated financial market

Stochastic Calculus

Weiner Process

$$* \; W_{t+1} = W_t + \varepsilon_t \quad (W_0 = 0, \;\; \varepsilon_t \sim N(0,1))$$

$$* \; W_{t+\Delta} = W_t + e_t \quad (W_0 = 0, \;\; e_t \sim N(0,\Delta), \; \Delta = \frac{1}{n})$$

Let $\Delta \to dt \;\; (\, i.e., \;\; n \to \infty \,)$

$$* \; W_{t+dt} = W_t + e_t \quad (W_0 = 0, \;\; e_t \sim N(0, dt))$$

$$\to dW = W_{t+dt} - W_t = e_t \sim N(0, dt)$$

Stochastic Calculus

Weiner Process

$$* \ dW = W_{t+dt} - W_t = e_t \sim N(0, dt)$$

$$= \varepsilon_t \sqrt{dt}, \quad (\varepsilon_t \sim N(0,1))$$

$$\rightarrow \begin{cases} E[dW] = 0 \\ E[(dW)^2] = dt \cdot E[\varepsilon_t^2] = dt \\ V[dW] = E[(dW)^2] - E^2[dW] = dt \\ V[(dW)^2] = E[(dW)^4] - E^2[(dW)^2] \\ \qquad = 3(dt)^2 - (dt)^2 = 0 \ \ (up\ to\ \ dt) \\ (dW)^2 \xrightarrow{dt \to 0} dt \end{cases}$$

Stochastic Calculus

Ito Process and Ito Lemma

* $dX = a(X_t,t)\,dt + b(X_t,t)\,dW$ (Ito Process)

$= a(X_t,t)\,dt + b(X_t,t)\,\varepsilon_t\sqrt{dt}, \quad (\varepsilon_t \sim N(0,1))$

* $G = G(X,t)$

$\rightarrow$

$$dG = \left[\frac{\partial G}{\partial X}dX + \frac{\partial G}{\partial t}dt\right] + \frac{1}{2!}\left[\frac{\partial^2 G}{\partial X^2}(dX)^2 + \frac{\partial^2 G}{\partial t^2}(dt)^2 + 2\frac{\partial^2 G}{\partial X \partial t}dXdt\right] + \cdots$$

$$= \left[\frac{\partial G}{\partial X}(a\,dt + b\,dW) + \frac{\partial G}{\partial t}dt\right] + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\left[b^2\,dt\right]$$

$$= \left[\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right]dt + \frac{\partial G}{\partial X}b\,dW$$

Stochastic Calculus

Geometric Brownian Motion

* $dX = a\,X_t\,dt + b\,X_t\,dW$ (Geometric Brownian Motion)
* $dS_t / S_t = \mu\,dt + \sigma\,dW$ (Assumption for stock price)

Let $Y = \ln S$

$\rightarrow$

$$dY = \left[\frac{\partial Y}{\partial S}\mu\,S + \frac{\partial Y}{\partial t} + \frac{1}{2}\frac{\partial^2 Y}{\partial S^2}\,\sigma^2\,S^2\right]dt + \frac{\partial Y}{\partial S}\sigma\,S\,dW = \left[\mu - \frac{1}{2}\sigma^2\right]dt + \sigma\,dW$$

$\rightarrow$

$$\begin{aligned}\ln S_T &= \ln S_0 + (\ln S_T - \ln S_0) = \ln S_0 + \int_0^T d\ln S_t \\ &= \ln S_0 + \int_0^T \left[\mu - \frac{1}{2}\sigma^2\right]dt + \int_0^T \sigma\,dW \\ &= \ln S_0 + \left[\mu - \frac{1}{2}\sigma^2\right]T + \sigma\,(W_T - W_0), \text{ where } (W_T - W_0 \sim N(0,T))\end{aligned}$$

Stochastic Calculus

Final Stock Value Distribution

$$* \ \ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]$$

$$\rightarrow * \ P_m(x) = \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}\left\{x - x(t) - \tau(\mu - \sigma^2/2)\right\}^2},$$

where $\left(S_T = e^x,\ S_0 = e^{x(t)}, \tau = T\text{-}t \right)$.

$$* \ E\left[S_T\right] = S_0\, e^{\mu T}$$

$$* \ V\left[S_T\right] = S_0^2\, e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)$$

Stochastic Calculus

Black-Scholes Equation for Hedge Portfolio

* $dS_t = \mu S\, dt + \sigma S\, dW$ (Assumption for stock price)
* $C = C(S,t)$ (Call Option)
* $V_H = N_S S + N_C C$ (Hedge Portfolio)

* $dV_H = rV_H dt$, where r: risk free rate. (Perfect Hedged Portfolio)

$\rightarrow$

$$dV_H = N_S dS + N_C (C_s dS + C_t dt + \frac{1}{2} C_{SS} \sigma^2 S^2) dt$$

$$= (N_S + N_C C_S) dS + N_C (C_t + \frac{1}{2} C_{SS} \sigma^2 S^2) dt$$

$\rightarrow N_S + N_C C_S = 0$, WLOG $N_S \equiv 1 \rightarrow N_C = -1/C_S$

$$\rightarrow r(N_S S + N_C C) = N_C (C_t + \frac{1}{2} C_{SS} \sigma^2 S^2)$$

$$\rightarrow C_t + rS \frac{\partial C}{\partial S} + \frac{1}{2} C_{SS} \sigma^2 S^2 = rC$$

Stochastic Calculus

- Solution for European Call Option

$$\begin{cases} \dfrac{\partial C}{\partial t} + rS\dfrac{\partial C}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} = rC \\ C(S,T) = \text{Max}\,(S\text{-}K,0) \\ \text{C}(0,t) = 0 \end{cases}$$

$$\rightarrow \quad C(\tau,S,K,r) = SN(d_+) - Ke^{r\tau}N(d_-)$$

where

$$\begin{cases} N(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\,dz\,, \quad d\pm = \dfrac{\ln(S/K) + (r \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\ \tau = T - t \end{cases}$$

Formalism of Quantum Mechanism

- Black-Scholes-Schrodinger Equation for Hedge Portfolio

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

$$Set\ S = e^x$$

$$\rightarrow \frac{\partial C}{\partial t} = \hat{H}_{BS} C, \quad \hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r$$

$$\rightarrow \hat{H}^+_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r \neq H_{BS}$$

$$(\ < f|O^+|g> \equiv < g|O|f>^* \ )$$

Formalism of Quantum Mechanism

Pricing kernel

$$p(x, y, \tau; x', y') \equiv p(x', y' \text{ at } T \mid x, y \text{ at } t),$$

where $\tau = T - t$, x:security price; y:volitility.

$$\rightarrow p(x, y, 0; x' y') = \delta(x' - x)\,\delta(y' - y)$$

$$\rightarrow C(\tau; x, y) = \int_{-\infty}^{+\infty} dx' dy' p(x, y, \tau; x', y') g(x' y')$$

with the final value condition $C(0, x, y) = g(x, y)$.

$$\rightarrow C(\tau; x) = \int_{-\infty}^{+\infty} dx' p_{BS}(x, \tau; x') g(x')$$

Formalism of Quantum Mechanism

Black-Scholes Pricing kernel

$$\frac{\partial C}{\partial t} = \hat{H}_{BS} C \rightarrow C(t;x) = e^{t\hat{H}} C(0;x)$$

$$<x|\frac{\partial}{\partial t}|C,t>=<x|\hat{H}|C,t>\rightarrow|C,t>=e^{t\hat{H}}|C,0>$$

$$|C,T>=e^{T\hat{H}}|C,0>=|g>\rightarrow|C,t>=e^{-(T-t)\hat{H}}|g>$$

$$\rightarrow C(t,x)=<x|C,t>=<x|e^{-\tau\hat{H}}|g>=\int_{-\infty}^{+\infty}dx'<x|e^{-\tau\hat{H}}|x'>g(x')$$

$$\rightarrow p(x,\tau;x')=<x|e^{-\tau\hat{H}}|x'>$$

$$\hat{H}=\hat{H}_{BS}=-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r$$

Formalism of Quantum Mechanism

- Black-Scholes Pricing kernel

$$< x | x' >= \delta(x - x') \rightarrow < x | p >= e^{ipx}, < p | x >= e^{-ipx}$$

$$< x | \hat{H}_{BS} | p >\equiv H_{BS}(p,x) < x | p >$$

$$= \left\{ \frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r \right\} e^{ipx} \quad (p = -i\frac{\partial}{\partial x})$$

$$p(x,\tau;x') =< x | e^{-\tau\hat{H}} | x' >= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} < x | e^{-\tau\hat{H}} | p >< p | x' >$$

$$= e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}\{x - x' + \tau(r - \sigma^2/2)\}^2}$$

Formalism of Quantum Mechanism

- Black-Scholes price for European call option

$$C(t,x) = \int_{-\infty}^{+\infty} dx' < x | e^{-\tau \hat{H}} | x' > g(x')$$

$$= \int_{-\infty}^{+\infty} dx' p(x,\tau,x') \mathrm{Max}(e^{x'} - K, 0)$$

$$= \int_{-\infty}^{+\infty} dx' e^{-\tau r} P_m(x') \mathrm{Max}(e^{x'} - K, 0)$$

$$= e^{-\tau r} E[\mathrm{Max}(S\text{-}K, 0)]$$

$$\rightarrow C(\tau, S, K, r) = SN(d_+) - Ke^{r\tau} N(d_-)$$

Black-Scholes Path Integral

$$p(x,\tau;x') = < x | e^{-\tau\hat{H}} | x' >$$

$$= \left(\prod_{i=1}^{N-1} \int dx_i \right) \prod_{i=1}^{N-1} < x_i | e^{-\varepsilon\hat{H}} | x_{i-1} >$$

$$< x_i | e^{-\varepsilon\hat{H}} | x_{i-1} > \equiv N_i(\varepsilon) e^{\varepsilon L(x_i; x_{i-1}; \varepsilon)}$$

$\rightarrow$

$$p(x,\tau;x') = \int_{BS} DX \; e^{S_{BS}}$$

where $S = \int L(x, dx/dt)\, dt, \quad L = -\frac{1}{2\sigma^2} \int_0^\tau dt \left(\frac{dx}{dt} + r - \frac{1}{2}\sigma^2 \right)^2$

Quantum description of Financial Market

Assume that a financial market consists of

1. Securities of types i=1,2,...,I;
2. Participants j=1,2,...,J.

At a given moment, a financial market state

$$|M\rangle = \sum_{n \in B} A_n |n\rangle$$

where

$$B := \left\{ \left| \left\{ x^j, \left\{ n_i^j(s) \geq 0, i=1,2,,\ldots,I \right\}, j=1,2,\ldots,J \right\} \right\rangle \right\}$$

$$N_i^j(s) = \int_0^s ds' n_i^j(s).$$

and $\sum_{n \in B} |A_n(t)| = 1$ holds at all times for an isolated market.

Quantum description of Financial Market

moving cash operators

$$\hat{c}^{+j}(s) = \exp(-is\hat{p}_j), \quad \hat{p}_j = -i\frac{\partial}{\partial x^j}$$

$\rightarrow$

$$\hat{c}^{+j}(s)|\{x^1, x^2, \ldots, x^j, \ldots, x^J\}>=|\{x^1, x^2, \ldots, x^j + s, \ldots, x^J\}>$$

satifying

$$\hat{c}^{+j}(s)\hat{c}^{+j}(s') = \hat{c}^{+j}(s+s')$$

$$\hat{c}^{+j}(-s) = \hat{c}^j(s), \hat{c}(0) = 1$$

and

$$\left[\hat{c}^{+j}(s), \hat{x}^k\right] = -s\delta^{jk}\hat{c}^{+j}(s).$$

Quantum description of Financial Market

- creating and destroying security operators

$$\left[\hat{a}_i^{\,j}(s),\hat{a}_l^{+k}(s')\right]=s\delta(s-s')\delta^{jk}\delta_{il}$$

$$\left[\hat{a}_i^{\,j}(s),\hat{a}_l^{\,k}(s')\right]=\left[\hat{a}_i^{+j}(s),\hat{a}_l^{+k}(s')\right]=0$$

$\rightarrow$

$\hat{a}_i^{\,j}(s)|0>=0$, for any i, j and $s \geq 0$.

$$|\{x^j,n_i^j(s)\}>\propto\prod_{j=1}^{J}\hat{c}^{+j}(x^j)\prod_{i=1}^{I}\prod_{j=1}^{J}\left(\hat{a}_i^{+j}(s)\right)^{n_i^j(s)}|0>$$

$$\hat{n}_i^j(s)\equiv\frac{1}{s}\hat{a}_i^{+j}(s')\hat{a}_i^{\,j}(s')$$

$\rightarrow$

$$\hat{n}_i^j(s)|\{x^j,n_i^j(s)\}>=n_i^j(s)|\{x^j,n_i^j(s)\}>$$

$$\hat{x}^j(s)|\{x^j,n_i^j(s)\}>=x^j(s)|\{x^j,n_i^j(s)\}>$$

Quantum description of Financial Market

● creating and destroying security operators

$$\left[\hat{a}_i^{\,j}(s), \hat{a}_l^{+k}(s')\right] = s\delta(s-s')\delta^{jk}\delta_{il} \quad \Rightarrow \begin{cases} \hat{b}_i^{+j}(s) \equiv \hat{a}_i^{+j}(s)\hat{c}_i^{\,j}(s) \\ \hat{b}_i^{\,j}(s) \equiv \hat{a}_i^{\,j}(s)\hat{c}_i^{+j}(s) \end{cases}$$

$$\left[\hat{a}_i^{\,j}(s), \hat{a}_l^{\,k}(s')\right] = \left[\hat{a}_i^{+j}(s), \hat{a}_l^{+k}(s')\right] = 0 \quad \rightarrow$$

$\rightarrow$

$$\hat{a}_i^{\,j}(s)|0> = 0, \text{ for any } i, j \text{ and } s \geq 0.$$

$$\left[\hat{b}_i^{\,j}(s), \hat{b}_l^{+k}(s')\right] = s\delta(s-s')\delta^{jk}\delta_{il}$$

$$|\{x^j, n_i^j(s)\}> \propto \prod_{j=1}^{J} \hat{c}^{+j}(x^j) \prod_{i=1}^{I}\prod_{j=1}^{J}\left(\hat{a}_i^{+j}(s)\right)^{n_i^j(s)} |0>$$

$$\left[\hat{b}_i^{\,j}(s), \hat{b}_l^{\,k}(s')\right] = \left[\hat{b}_i^{+j}(s), \hat{b}_l^{+k}(s')\right] = 0$$

$$\hat{n}_i^j(s) \equiv \frac{1}{s}\hat{a}_i^{+j}(s')\hat{a}_i^{\,j}(s')$$

$$\left[\hat{b}_i^{\,j}(s), \hat{p}^k\right] = \left[\hat{b}_i^{+j}(s), \hat{p}^{+k}\right] = 0$$

$\rightarrow$

$$\hat{n}_i^j(s)|\{x^j, n_i^j(s)\}> = n_i^j(s)|\{x^j, n_i^j(s)\}>$$

$$\hat{x}^j(s)|\{x^j, n_i^j(s)\}> = x^j(s)|\{x^j, n_i^j(s)\}>$$

$$\left[\hat{b}_i^{+j}(s), \hat{x}^k\right] = s\delta^{jk}\hat{b}_i^{+j}(s)$$

$$\left[\hat{b}_i^{\,j}(s), \hat{x}^k\right] = -s\delta^{jk}\hat{b}_i^{\,j}(s)$$

Quantum description of Financial Market

timporal market evolution

${}_t\langle M | M \rangle_t = 1$ for an isolated market

$\rightarrow$

$$|M\rangle_{t'} = \hat{U}(t',t)|M\rangle_t \text{ where } \hat{U}(t',t) \text{ is an unitary operator.}$$

$\rightarrow$

$$\hat{H}(t) = i\frac{\partial}{\partial t'}\hat{U}(t',t)|_{t'=t} \text{ which is a Hermitian Hamiltonian.}$$

$\rightarrow$

$$\hat{U}(t',t) = 1 - \int_t^{t'} dt_1 \hat{H}(t_1) + (-i)^2 \int_{t_1}^{t'} dt_1 \int_t^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} T\left[-i\int_t^{t'} d\tau \hat{H}(\tau)\right]^n$$

$$\rightarrow i\frac{\partial}{\partial t}|M\rangle_t = \hat{H}(t)|M\rangle_t$$

Quantum description of Financial Market

Hamiltonian for cash flow

Hamiltonian Equations:

$$\frac{dx(t)}{dt}=\frac{\partial H(x,p,t)}{\partial p},\quad \frac{dp(t)}{dt}=-\frac{\partial H(x,p,t)}{\partial x}$$

and

$$\frac{dx^j(t)}{dt}\equiv r^j(t)x^j(t)\ \rightarrow H^j(x^j,p_j;t)=r^j(t)x^j p_j$$

$\rightarrow$

$$\hat{H}(t)=\sum_{j=1}^{J}H^j(t),\quad \hat{H}^j(t)=\frac{r^j(t)}{2}(\hat{x}^j\hat{p}_j+\hat{p}_j\hat{x}^j)$$

up to some function $V(\{x^k\})$.

Quantum description of Financial Market

● Hamiltonian for trading securities

Please see "Quantum Finance"

from

arXiv:physics/0203006v2 [physics.soc-ph]

Conclusion

This talk leaves many questions need to be clarified. Is it a really a quantum system, or we just use quantum formalism to describe the quantitative finance. However, stochastic models usually ignore the market fluctuations and the phenomena of coherent effects which both can be represented using such a quantum formalism.