Conformal Killing-Yano two-forms and Near-horizon geometry[1]

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CYCU HEP seminar
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[1] based on work with Yoshihiro Mitsuka arXiv:1110.3872

Outline

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Symmetries

On a curved manifold with metric g, we are interested in field equations:

- Klein-Gordon $(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\Phi = 0$
- Maxwell $\nabla^\mu F_{\mu\nu} = \partial_{[\mu}F_{\nu\rho]} = 0$
- Dirac $(i\gamma^\mu\nabla_\mu + m)\Psi = 0$

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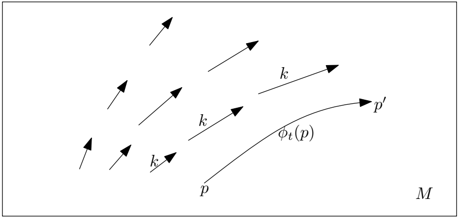
What about spacetime symmetries?

given by Killing vectors $=$ isometries of the metric

Reminder on diffeomorphisms

From flows to vector fields and back:



One can integrate a vector field k (infinitesimal diffeomorphism) to a flow (one-parameter finite diffeomorphism) $\phi_t : M \to M$,

$$\phi_0 = \mathrm{Id} \text{ and } \phi_t \circ \phi_{t'} = \phi_{t+t'}$$

and conversely, with

$$\frac{\mathrm{d}}{\mathrm{d}t} x^\mu \left(\phi_t(p)\right)\Big|_{t=0} = k^\mu$$

Isometries

Killing vectors: those infinitesimal diffeomorphisms that leave the metric invariant

$$\delta g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\rho\mu}$$
$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

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Equivalently:

$\nabla_\mu \xi_\nu = A_{\mu\nu}$ is a two-form, antisymmetric in $[\mu, \nu]$.

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E.g., Klein-Gordon equation with $[\mathcal{L}_\xi, \nabla] = 0$ and $[\mathcal{L}_\xi, g] = 0$ or $[\mathcal{L}_\xi, d] = 0$ and $[\mathcal{L}_\xi, *] = 0$:

$$\left(- * d * d - m^2\right) \Phi = 0 \xrightarrow{\mathcal{L}_\xi} \left(- * d * d - m^2\right) (\mathcal{L}_\xi \Phi) = 0$$

Symmetry operators of Dirac

However there might be *more* "spacetime" symmetries. Dirac equation is

$$(i\gamma^\mu \nabla_\mu + m)\Psi = 0$$

Spinor endomorphisms are mixed-degree differential forms*

$$P_0 = \sum_{p=0}^{d} \frac{1}{p!} P_0^{[p]}{}_{\mu_1 \cdots \mu_p} \gamma^{\mu_1 \cdots \mu_p} \in \mathrm{End}(S)$$

First order operators are given by $L = P_1 + P_0$ with

$$P_1 = \sum_{p=0}^{d+1} \frac{1}{n!} P_1^{[p]}{}_{\mu_1 \cdots \mu_{p-1}|\mu_p} \gamma^{\mu_1 \cdots \mu_{p-1}} \nabla^{\nu_p}$$

When does $\not\nabla L = R \not\nabla$ (R-commute) and when does $R = 1$?

Conformal Killing-Yano as symmetries

The most general first-order operator[2] that R-commutes with $\not\nabla$ is $L = L_K + (\text{staff})\not\nabla$ where

$$L_K = \gamma^\mu K \nabla_\mu + \frac{p}{p+1} dK - \frac{d-p}{d-p+1}\delta K$$

and K is a Conformal Killing-Yano p-form:

$$\nabla_\mu K_{\nu_1\cdots\nu_p} = A_{\mu\nu_1\cdots\nu_p} + p\, g_{\mu[\nu_1} B_{\nu_2\cdots\nu_p]}\ .$$

It commutes $R = 1$ if $B = 0$ and either

- Dimension d is even.
- Dimension d is odd and degree p is odd.

[2] Benn,Charlton hep-th/9612011

Conformal Killing-Yano as generalization

ξ is a Killing vector ($\mathcal{L}_\xi g = 0$):

$$\nabla_\mu \xi_\nu = A_{\mu\nu}$$

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Conformal Killing-Yano definition is equivariant under

1. Homotheties $g \mapsto e^{2\Lambda} g \Rightarrow K \mapsto e^{(p+p!)\Lambda} K$
2. Hodge duality $K \mapsto *K \Rightarrow (A, B) \mapsto ((-1)^{p-1} * B, (-1)^p * A)$

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I.e., they are conformally invariant

More on CKY and symmetries

(Conformal) Killing-Yano p-forms - (C)KY - give

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- Constants of (lightlike) geodesic motion

$$\dot{\sigma}^\mu \nabla_\mu \dot{\sigma}^\nu = 0 \Rightarrow C = \dot{\sigma}^\mu \dot{\sigma}^\nu \left(K_{\mu\nu_1 \cdots \nu_{p-1}} K_\nu{}^{\nu_1 \cdots \nu_{p-1}} \right) \text{ with } \dot{\sigma}^\mu \nabla_\mu C = 0$$

or constant tensors along geodesic $\dot{\sigma}^\mu \nabla_\mu (\dot{\sigma}^\nu K_{\nu\nu_1 \cdots \nu_{p-1}}) = 0$

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$$L^2 = \nabla_\mu (K^{\mu\nu_2\cdots\nu_p} K^\nu{}_{\nu_2\cdots\nu_p} \nabla_\nu)$$

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- action geometric Killing spinors (generalizing spinorial Lie derivative)
- Uniqueness of Kerr, geometric structure, ...

Killing transport of isometries

What kind of beast is an equation like "$\nabla_\mu \xi_\nu = A_{\mu\nu}$"?

$$\nabla_\mu A_{\nu\rho} - \nabla_\nu A_{\mu\rho} = R_{\mu\nu\rho\sigma}\xi^\sigma$$
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Killing vectors are parallel under a connection D on $\Lambda^1 \oplus \Lambda^2$

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi - i_X A \\ \nabla_X A - R(X,\xi) \end{pmatrix} = 0$$

Maximum number of solutions $d + \frac{d}{2}(d-1)$ for spheres, flat, (A)dS, $\mathcal{H}$, and discrete quotients thereof.

Killing transport of CKY

Similarly CKY p-forms are in one-to-one correspondence with parallel section

$$K + A + B + C \in \Lambda^p \oplus \Lambda^{p+1} \oplus \Lambda^{p-1} \oplus \Lambda^p$$

under a connection[3] D.

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under a connection[3] D.
E.g., for $p = 2$

$$\begin{aligned}
\nabla_\mu K_{\nu\rho} &= A_{\mu\nu\rho} + g_{\mu\nu} B_\rho - g_{\mu\rho} B_\nu \\
\nabla_\mu A_{\nu_1\nu_2\nu_3} &= -\frac{3}{2} R_{[\nu_1\nu_2|\mu}{}^\sigma K_{\sigma|\nu_3]} - \frac{3}{4} g_{\mu[\nu_1} C_{\nu_2\nu_3]} \\
\nabla_\mu B_\nu &= \frac{1}{4} C_{\mu\nu} - \frac{1}{2}\frac{1}{d-2}(R_{\sigma\mu} K^\sigma{}_\nu + R_{\sigma\nu} K^\sigma{}_\mu) \\
\nabla_\mu C_{\nu_1\nu_2} &= \ldots
\end{aligned}$$

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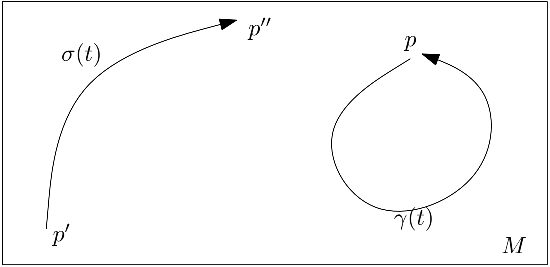
CKY transport and holonomy

Given g, one can try to solve:



$$D_\mu \mathcal{E} = 0 \Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t} + \dot{\sigma}^\mu(t) \mathcal{A}_\mu(\sigma(t)) \right) \mathcal{E}(t) = 0$$

Transport $P_\sigma(\mathcal{E}_{p'}) = \mathcal{E}_{p''}$. Holonomy on solution

$$P_\gamma \mathcal{E}_p = \mathcal{E}_p$$

for closed loops γ. Solutions are *singlets* under holonomy of D.

Einstein d=4, p=2

For $p = 2$ note

$$\nabla_\mu B_\nu = \frac{1}{4} C_{\mu\nu} - \frac{1}{2}\frac{1}{d-2}(R_{\sigma\mu}K^\sigma{}_\nu + R_{\sigma\nu}K^\sigma{}_\mu)$$

If the background is Einstein then B is a Killing one-form.
Also CKY is equivariant under Hodge duality,
$K \mapsto *K \Rightarrow (A, B) \mapsto (-*B, *A)$. So for Einstein, $d = 4$, $p = 2$:

both B and $*A$ are Killing one-forms

The right-hand side of

$$\nabla_\mu K_{\nu\rho} = A_{\mu\nu\rho} + g_{\mu\nu}B_\rho - g_{\mu\rho}B_\nu$$

is in terms of $2N$ unknown *constants*, where $N \leq \frac{d}{2}(d+1)$ Write this as

$$K \overset{\pi}{\mapsto} (\tilde{\xi}, \xi)$$

CKY under isometries

Assume a Killing vector k and a CKY p-form K:

$$
\begin{aligned}
0 =&\mathcal{L}_k \left(\nabla_X K - i_X A - X^\flat \wedge B\right) \\
=&\nabla_{[k,X]} K - i_{[k,X]} A - [k, X]^\flat \wedge B \\
&+ \nabla_X \mathcal{L}_k K - i_X \mathcal{L}_k A - X^\flat \wedge \mathcal{L}_k B \\
=&\nabla_X \mathcal{L}_k K - i_X \mathcal{L}_k A - X^\flat \wedge \mathcal{L}_k B \ .
\end{aligned} \tag{1}
$$

Therefore, CKY two-forms form a representation under the isometry algebra of the metric:

$$
K \overset{\pi}{\mapsto} (\tilde{\xi},\, \xi) \Rightarrow \mathcal{L}_k K \overset{\pi}{\mapsto} (\mathcal{L}_k \tilde{\xi},\, \mathcal{L}_k \xi)
$$

We can fix most of the right-hand side A and B by using the action of the isometries.

Outline

Uniqueness theorems

- A black hole: a region of spacetime from which nothing, not even light, can escape.
- Stationary metrics: no time dependence $\mathcal{L}_{\partial_t} g = 0$
- No-hair theorems/conjectures and uniqueness theorems: stationary black holes have only a finite number of parameters = charges
- For Einstein solutions[4], charges are: mass, angular momentum and NUT charge (if asymptotically *only locally* AdS/flat) $\int_{S^2} \pi^* d(\partial_t{}^\flat)$
- (local) uniqueness[5] of Kerr from a closed CKY 2-form ($A = 0$)

[4] Israel '67, Carter '71, Robinson '75

[5] Houri et al. 0708.1368, Krtous et al. 0804.4705

Kerr metric

Kerr's solution of $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{\ell^2}g_{\mu\nu} = 0$

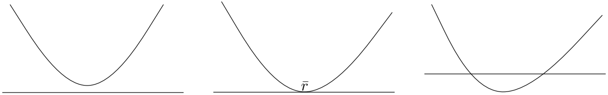
$$\mathrm{d}s^2 = -\frac{\Delta_r}{r^2+y^2}\left(\mathrm{d}t + y^2\mathrm{d}\psi\right)^2 + \frac{\Delta_y}{r^2+y^2}\left(\mathrm{d}t - r^2\mathrm{d}\psi\right)^2 + \frac{r^2+y^2}{\Delta_r}\mathrm{d}r^2 + \frac{r^2+y^2}{\Delta_y}\mathrm{d}y^2 \ ,$$

$$\Delta_r = (1+\frac{r^2}{\ell^2})(r^2+a^2) - 2Mr \ , \quad \Delta_y = (a^2-y^2)(1-\frac{y^2}{\ell^2}) + 2Ly \ .$$

- Mass M
- angular parameter a
- NUT charge L
- cosmological const. $-3/\ell^2$

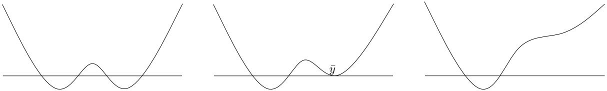
- $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \overset{r^2+y^2\to 0}{\longrightarrow} \infty$
- signature $(-,+,+,+)$
- range of parameters
- periodicities $ds^2| = dr^2 + r^2 d\theta^2$

Graphs of metric functions



(a) $M < M_*$ (b) $M = M_*$ (c) $M > M_*$

Figure: $\Delta_r(r)$ in Kerr-AdS for fixed a.



(a) $|L| < L_*$ (b) $|L| = L_*$ (c) $|L| > L_*$

Figure: $\Delta_y(y)$ in Kerr-AdS for fixed a.

So we need $M \geq M_*(a)$ and $|L| \leq L_*(a)$ to shield singluarities.

Physical parameters

For negative comsmological constant

- Given mass M, the black hole cannot over-rotate $|a| \leq M_*(a)$
- Given rotation parameter a, the NUT charge cannot be too large $|L| \leq L_*(a)$

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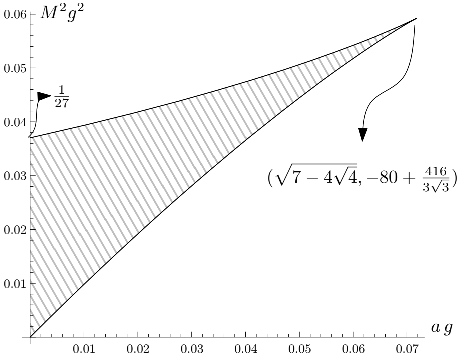
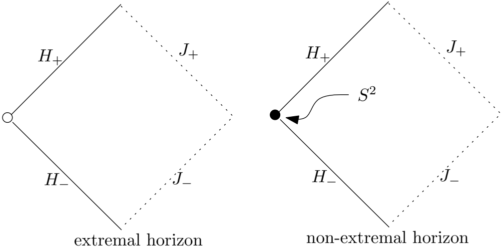
For positive cosmological constant, $\ell^2 \to -1/g^2$:



Extremality

At extremality $M = M_*(a)$:

- double root $\Delta_r(\bar{r}) = 0$
- Hawking temperature $T_H = 0$
- Future and past horizon do not intersect/bifurcate



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When NUT $L = 0$, ψ is periodic. When NUT $L \neq 0$, coordinates (t, ψ) describe *torus* fibers, because the two roots of Δ_y impose different periodicities close to each root

$$ds^2| \approx dr^2 + r^2 d\theta^2$$

At extremality $L = L_*(a)$:

- double root $\Delta_y(\bar{y}) = 0$ (e.g. on the right)
- only one periodicity again (from the root on the left) and an infinite throat at the double root $y = \bar{y}$.

Similar to extremal mass, with r and y exchanged.

Geroch's results on spacetime limits

Definition

Take a smooth family of spacetimes $(M_\epsilon, g_\epsilon, \epsilon)$ for $\epsilon > 0$.
If $\lim_{\epsilon\to 0} g_\epsilon = g_0$ exists, it is a "spacetime limit in the family".

Definition

If $f_\epsilon : M_1 \to M_\epsilon$ for $\epsilon > 0$ is an isometry, then g_0 is a "limit of the metric g_1".

If g_0 is not isometric to g_1 then we have something new!

Theorem (Geroch '69)

The kernel of a connection D does not reduce its dimension under a spacetime limit. A holonomy result.

An example of a metric limit is the near-horizon limit.

Near-horizon limits

Extremal black holes in gaussian coordinates at the horizon[6] $r = 0$

$$g = r^2 F(r)\, du^2 + G(r)\, dr\, du + \cdots$$

[6] Racz, Wald '92

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The diffomorphism with $\epsilon > 0$

$$r \mapsto r' = r/\epsilon \; , \quad u \mapsto u' = \epsilon u$$

a) preserves the horizon and b) for $r' \in [0, 1]$ zooms in with ϵ.

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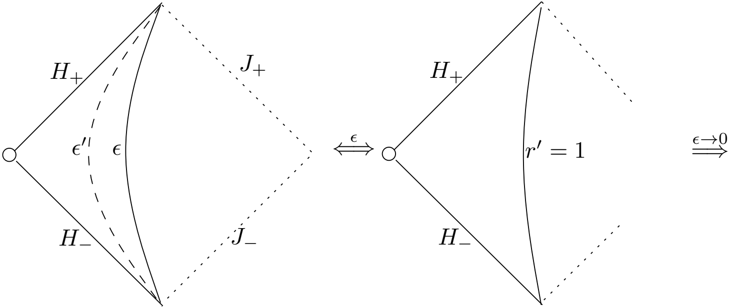

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The output of the near-horizon limit $\epsilon \to 0^+$ is

1. a new metric g_0
2. a Killing horizon $r' = 0$ (but not a BH - not asympt. to AdS/flat)

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For non-extremal black holes

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For non-extremal black holes, one can take $r_+ \to r_-$ as $\epsilon \to 0^+$, but ...

[6] Racz, Wald '92

Outline

The near-horizon limit (locally)

Taking the limit $\epsilon \to 0^+$ of extremal Kerr:

$$\mathrm{d}\bar{s}^2 = \Omega^2(y)\left(-x^2\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2(y)(\mathrm{d}u + x\,\mathrm{d}\tau)^2\right) + \frac{\bar{r}^2 + y^2}{\Delta_y}\mathrm{d}y^2 \ ,$$

with

$$\Omega^2 = \beta^2(\bar{r}^2 + y^2) \ , \qquad \Omega^2\Lambda^2 = \frac{\Delta_y}{\bar{r}^2 + y^2}4\bar{r}^2\beta^4 \ , \qquad \beta(\bar{r}, L).$$

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Similarly, for the polar limit $\epsilon \to 0^+$:

$$\mathrm{d}\bar{s}'^2 = \Omega^2(r)\left(+x^2\mathrm{d}\bar{\psi}^2 + \frac{\mathrm{d}x^2}{x^2} - \Lambda^2(r)\left(\mathrm{d}u + x\,\mathrm{d}\bar{\psi}\right)^2\right) + \frac{\bar{y}^2 + r^2}{\Delta_r}\mathrm{d}r^2 \ ,$$

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The near-horizon limit (periods)

The near-horizon limit of extremal Kerr:

$$d\bar{s}^2 = \Omega^2(y)\left(-x^2 d\tau^2 + \frac{dx^2}{x^2} + \Lambda^2(y)(du + x\, d\tau)^2\right) + \frac{\bar{r}^2 + y^2}{\Delta_y} dy^2 \ ,$$

We find

- the near-horizon limit is well-defined only for $L = 0$, otherwise the torus lattice degenerates. In this first case $u = u + 2\pi T(\bar{r}, L)$.
- The polar limit is always well-defined, with $u = u + 2\pi T(\bar{y}, M)$.

However, there are reasons to consider the NHEK metric with two parameters...

NHEK as a general solution

Assume a metric

$$\mathrm{d}s^2 = \Omega^2(y)\left[-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2(y)\left(\mathrm{d}u + x\,\mathrm{d}\tau\right)^2\right] + F^2(y)\mathrm{d}y^2.$$

and fix $F = 1$. Einstein's equations become

$$\dddot{\Lambda} = f(\Lambda, \dot{\Lambda}, \ddot{\Lambda}) \text{ and } \Omega = g(\Lambda, \dot{\Lambda}, \ddot{\Lambda})$$

Gauge $F = 1$ is preserved by $y \mapsto y + c$. So most general solution depends on *two* integration constants:

$\Rightarrow$ solution is locally the NHEK metric - but with $\bar{r}^2, L \in \mathbb{R}$!

AdS_4 is included with $F = \ell$, $\Lambda^2 = 1$ and $\Omega^2 = \ell^2 \cosh^2\left(y/\ell^2\right)$ - but with $\bar{r}^2 = -1$!

Deformed$^+$ AdS in NHEK

At fixed y the NHEK becomes

$$\mathrm{d}s^2| = \Omega^2(y)\left[-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2(y)\left(\mathrm{d}u + x\,\mathrm{d}\tau\right)^2\right] + \cancel{F^2(y)\mathrm{d}y^2}.$$

or

$$\mathrm{d}s^2| = \Omega^2\left(-\theta^0\otimes\theta^0 + \theta^1\otimes\theta^1 + \Lambda^2\theta^2\otimes\theta^2\right)$$

where the θ's are the right-invariant one-forms of $\mathrm{SL}(2,\mathbb{R})$.

For $\Lambda = 1$, $ds^2|$ *is* the "round" metric on $\mathrm{SL}(2,\mathbb{R})$, which gives Anti-de Sitter in $d = 3$.

Deformed^{+} AdS isometries

AdS_3 is $\mathrm{SL}(2,\mathbb{R})$ with metric:

$$g = \Omega^2(-\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2) \ ,$$

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l_a: right-invariant vector fields that generate the left-action
r_a: left-invariant vector fields that generate the right-action

$$\mathcal{L}_{r_a}\theta^b = 0 \qquad\qquad \mathcal{L}_{l_a}\theta^b = \epsilon_a{}^b{}_c\theta^c$$

Killing algebra of AdS$_3$ is

$$\langle l_a\rangle \oplus \langle r_a\rangle = \mathrm{sl}(2,\mathbb{R})_L \oplus \mathrm{sl}(2,\mathbb{R})_R = \mathrm{so}(1,2) \oplus \mathrm{so}(1,2)$$

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NHEK has isometries $\mathrm{sl}(2,\mathbb{R}) \oplus \mathbb{R}$

Outline

Kerr/CFT

- We have seen that the near-horizon limit has more isometries than the Kerr black hole (enhancement) from $\langle \partial_t \rangle \oplus \langle \partial_\psi \rangle$ to $\mathrm{sl}(2, \mathbb{R}) \oplus \langle l_2 \rangle$
- Geroch says CKY 2-forms also do not reduce
- It is *conjectured* that the Kerr black hole is dual to a CFT2
  - Entropy can be written as that of a CFT2
  - the right Virasoro is related to the r_a
  - Bulk correlators of scalars are approximately CFT2 correlators
  - Some discussion on *hidden* symmetry

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Questions for the NHEK

- Is there a second $\mathrm{sl}(2,\mathbb{R})$?
- Do the conformal Killing-Yano two-forms enhance?

Reminder

We have seen that a CKY 2-form in a $d = 4$ Einstein spacetime:

0. Definition

$$\nabla_\mu K_{\nu\rho} = A_{\mu\nu\rho} + g_{\mu\nu} B_\rho - g_{\mu\rho} B_\nu$$

1. Both $\tilde{\xi} = B$ and $\xi = *A$ are Killing one-forms/vectors

$$K \overset{\pi}{\mapsto} (\xi,\, \tilde{\xi})$$

2. The hodge dual is also a CKY two-form with

$$*K \overset{\pi}{\mapsto} (\tilde{\xi},\, -\xi)$$

3. If K is a CKY two-form and k is a Killing vector, then

$$\mathcal{L}_k K \overset{\pi}{\mapsto} (\mathcal{L}_k \xi,\, \mathcal{L}_k \tilde{\xi})$$

Fixing r Part 1

The NHEK is known to have a CKY two-form K_p such that

$$*K_p \overset{\pi}{\mapsto} (l_2,\, 0) \text{ and } K_p \overset{\pi}{\mapsto} (0,\, -l_2).$$

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Killing vectors are $\langle r_a \rangle \oplus \langle l_2 \rangle$ and the CKY defining equation is $\mathbb{R}$-linear:
If there are more CKY 2-forms, then there is a CKY 2-form K such that

$$K \overset{\pi}{\mapsto} (r,\, r'),$$

where $r = A\, r_0 + B\, r_1 + C\, r_2$ and similar for r'.

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If r is colinear with r', use Hodge duality. If r is not colinear with r', then consider $\mathcal{L}_r r' \neq 0$. In either case:

$$K \overset{\pi}{\mapsto} (r,\, 0)$$

Fixing r Part 2

Action of $\mathrm{sl}(2,\mathbb{R})$ on the r_a is irreducible:

$$r = A\,r_0 + B\,r_1 + C\,r_2 \overset{\mathrm{so}(1,2)}{\mapsto} \begin{cases} \pm\sqrt{B^2 + C^2 - A^2}\,r_2 \\ \pm\sqrt{-B^2 - C^2 + A^2}\,r_0 \\ \pm r_0 \pm r_2 \end{cases}$$

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If there are more than two CKY two-forms, then there are eight. Three of them are Killing-Yano ($B = 0$)

$$K_a \overset{\pi}{\mapsto} (r_a,\, 0)$$

and another three are closed CKY ($A = 0$)

$$*K_a \overset{\pi}{\mapsto} (0,\, -r_a).$$

Fixing K Part 1

We have completely fixed the right-hand side of

$$\nabla_\mu K_{\nu\rho} = g_{\mu\nu} B_\rho - g_{\mu\rho} B_\nu$$

with $B_\mu = (r_a)_\mu$.

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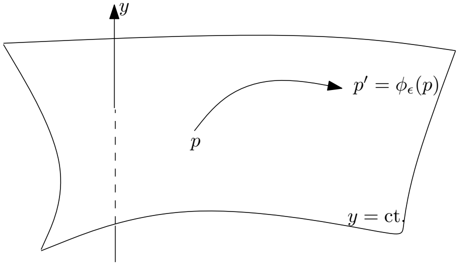
There are $3 \times \frac{4}{2}(4-1) = 2 \times 3^2 = 18$ unknown functions of (x, τ, u, y) on the left-hand side. We can partially fix them by active transformations:

$$\left.\begin{aligned} K_a &\overset{\pi}{\mapsto} (r_a,\, 0) \\ e^{\epsilon \mathcal{L}_r} K_a &\overset{\pi}{\mapsto} (e^{\epsilon \mathcal{L}_r} r_a,\, 0) \\ \phi_\epsilon^* K_a &\overset{\pi}{\mapsto} ({S_a}^b(\epsilon) r_b,\, 0) \end{aligned}\right\} \Rightarrow \left. K_a \right|_{\phi_\epsilon(p)} = {S_{\epsilon a}}^b {\phi_{-\epsilon}}^* \left. K_b \right|_p \ .$$

The right action acts transitively on y-constant surfaces.

$\Rightarrow$ 18 functions of y.

Fixing K Part 2



At any point p of a slice y we have 18 coefficients. At any other point of the *same slice*, K is given by:

$$K_a = \theta^b(r_a)\Big(H(y)_{bc}\frac{1}{2}\epsilon^c{}_{de}\hat{\theta}^d \wedge \hat{\theta}^e + G(y)_{bc}\mathrm{d}y \wedge \hat{\theta}^c\Big)\ ,$$

where $\hat{\theta}^a$ are right-invariant.
H_{ab} and G_{ab} are $2 \times 3^2 = 18$ functions of y alone: easy to solve.

Equations to solve

The defining equation

$$\nabla_\mu K_{\nu\rho} = g_{\mu\nu} B_\rho - g_{\mu\rho} B_\nu$$

becomes first-order differential equations in y. The y-dependence is solved and we get 9 linear equations for the 4 non-zero H_{ab} and G_{ab}, e.g.

$$\begin{aligned} H_{22} + H_{00}\frac{1}{2}\Lambda + G_{00}\dot{\Omega}F^{-1} &= -\Omega^2 \\ -H_{11} - H_{00}\left(1 - \frac{1}{2}\Lambda^2\right) - G_{00}(\dot{\Omega} + \dot{\Lambda})F^{-1} &= \Omega^2\Lambda \\ &\vdots \end{aligned}$$

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⇒ Unless the NHEK solution is precisely AdS$_4$, there is no solution.

Summary

We saw:

- definition, usefulness, and properties of conformal Killing-Yano p-forms
- the near-horizon limit and a motivation to study it
- when there is enough isometries, how to reduce the CKY equation

In the work with Y.Mitsuka:

- we derived the connection D of CKY p-forms in d dimensions
- commented on the near-horizon: the limit (periodicities) is not well-defined for non-zero NUT
- introduced a polar extremal limit
- for the NHEK and polar limit geometry: either one and only one Killing-Yano two-form, or AdS_4 in which case $2 \times \frac{5}{2}(5-1) = 20$ CKY 2-forms

Outline

Not the end

Possible extensions:

- adding flux to Einstein[7]
- higher dimensions and degrees
- quest for non-trivial CKY algebra*
- How do the hidden spacetime symmetries realize in supergravity?

[7] Cariglia et al. 1102.4501

Geometric Killing spinors

Spinor inner product, e.g. in $d = 4$

$$\bar{\epsilon}_1 \epsilon_2 = -\bar{\epsilon}_2 \epsilon_1 \text{ and } \bar{\gamma}_\mu = -\gamma_\mu$$

Geometric Killing spinor

$$\nabla_\mu \epsilon = \lambda \gamma_\mu \epsilon$$

so that

$$\nabla_\mu \left(\bar{\epsilon}_1 \gamma_{\nu_1 \cdots \nu_p} \epsilon_2 \right) = \lambda \bar{\epsilon}_1 [\gamma_{\nu_1 \cdots \nu_p}, \gamma_\mu] \epsilon_2$$

is antisymmetric for p odd

Freund-Rubin backgrounds, e.g. $\text{AdS}_4 \times S^7$, have supergravity Killing spinors the tensor of geometric Killing spinors

$$\epsilon = \epsilon_{\text{AdS}} \otimes \epsilon_S$$

However, symmetric square of supergravity field variations only gives the Killing vector plus trivial gauge shifts. Consistent reductions?